Notes on Cosmology

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1 Metric of a 2D spherical shell

We see in various books on Cosmology that distance (metric) can be specified on a 2D spherical shell in the form:

$$ds^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\phi^2 \tag{1}$$

But hardly any of them have a good explanation on exactly what it means; they just say that $K = 0$ is a plane (obvious), but don't elaborate on how $K = 1$ shows the equation of a spherical shell! Later they extend it to a 3D spherical shell in a 4D space. The thing that was particularly hard for me to understand the first time I confronted this metric was that: *What are r?* It isn't the radius since we are talking about a spherical shell so the distance from the center (the radius) should not change. Here is the explanation:

To understand this relation first think of a flat, 2D surface; Fig.1. Two of the most common ways to define the points on a 2D surface is the (x, y) cartesian coordinates or the (r, ϕ) polar coordinates, both centered on A. The most general infinitesimal change in these can be to: $(x + dx, y + dy)$ or $(r + dr, \phi + d\phi)$, based on which coordinates you use. dx and dy are along the x and y coordinates and not drawn in Fig.1.

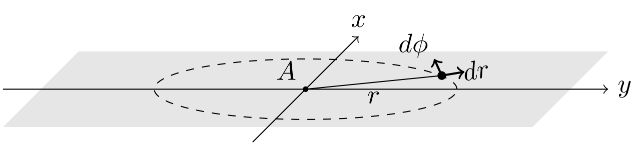

Figure 1: Polar coordinates on a 2D surface

Using the Pythagorean theorem, we can simply see that the distance that results from such a change of position in the 2D cartesian coordinates is:

$$ds^2 = dx^2 + dy^2 \tag{2}$$

and for the polar coordinates we can write:

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{3}$$

Which is just eq.1 with $K = 0$.

As I explained above, this problem is later generalized to a 3D surface. So it is very important to grasp the concept in this much simpler level rather than the next, so imagine your self as a 2D creature on this plane, completely unable to physically imagine a third dimention, but able do mathematics.

How could such a creature on the point A know if the flat and easy to understand world *it thinks it lives in* (Fig.1) is correct? The world he lives in might equally be a locally flat but globally curved surface like Fig.2. The answer to this question but for a 3D being (us) is the whole purpose to this discussion in all the books and one of the main questions still alive in Cosmology. So here we want to give that creature the tools to discover the curvature of the universe it lives in.

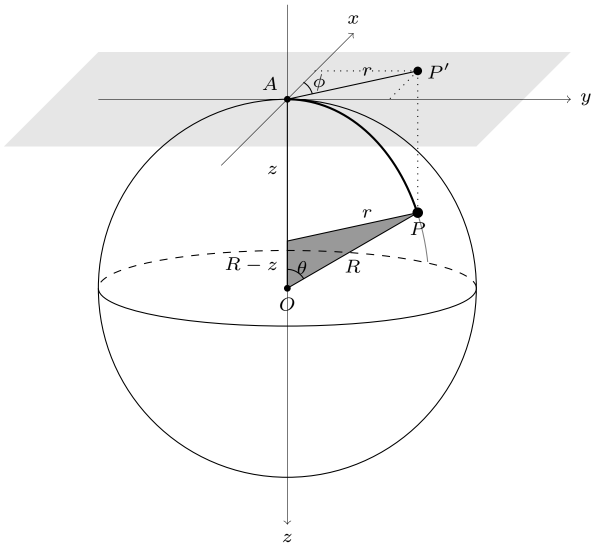

Figure 2: Polar coordinates on a 2D surface

To do that, we assume a flat, third cartesian coordinate that is orthogonal to the first two[1]; z. We take the 2D observer's position to be point A of Fig.2, that is also the only point that the two surfaces (flat and curved) share.

In an infinite 3D catesian space, distance can be expressed as:

$$ds^2 = dx^2 + dy^2 + dz^2$$

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[1]The 2D creature is unable to physically imagine this dimention, but can do the same mathematical process.

or

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2 \tag{4}$$

but we are not dealing with such a space, we have a 2D spherical shell, so let's constrain the three cartesian coordinates so we can find ds^2 only on the spherical shell. We take an arbitary point P for this purpose on the spherical shell, it's image on the flat plane is shown as P' which is a distance r from A. From the dark triangle in Fig.2 we see that:

$$\sin\theta = \frac{r}{R} \qquad \cos\theta = \frac{R-z}{R}$$

using $\sin^2\theta + \cos^2\theta = 1$, we get: $z^2 - 2Rz + r^2 = 0$, the solution of this can be found to be:

$$z = R\left(1 \pm \sqrt{1 - \frac{r^2}{R^2}}\right) \tag{5}$$

To see how z will change with a small change in r we find the derivative to be:

$$dz = \frac{\mp r}{R}\left(\frac{1}{\sqrt{1 - r^2/R^2}}\right) dr \tag{6}$$

We can't take both answers, so we limit outselves to one hemisphere and take the + sign. Placing this value of dz in eq.4, we find:

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\phi^2 \tag{7}$$

So we see that if we take $K = 1/R^2$, then the two equations 1&7 will be equal. When the books say $K = 1$ is a spherical shell, they are talking of a spherical shell with radius $R = 1$. They then multiply an $a(t)$ to the whole right side of the equation to set the scale:

$$ds^2 = a^2(t)\left(\frac{dr^2}{1 - r^2/R^2} + r^2 d\phi^2\right) \tag{8}$$

Very important: In deriving this equation we only used half of the sphere in choosing either of the + or − signs from eqs.5&6. So this metric is only valid for a hemisphere of a 2D spherical shell, so when we use this metric to find properties of the whole space we have to multiply it by 2.

To get a clear understanding of this equation and understand the relation between dr and ds, we should place our selves on the spherical shell; have a look at Fig 3. It is clear that dr is the differential distance on a *flat* surface; for a path between A to B', which is the *image* of B on the flat plane. ds is the differential distance between A and B along the surface of the sphere. So, *dr and $d\phi$ in eqs.7&8, are still those of the flat polar coordinates*; the gray plane in Fig.3

Lets solve eq.7 for the condition in Fig.3 to complete the explanation: Assume that the radius of this sphere is R and the distance between A and B' to also be R. It is evident that $d\phi = 0$ because ϕ doesn't change in this movement, so eq. 7 would lead to:

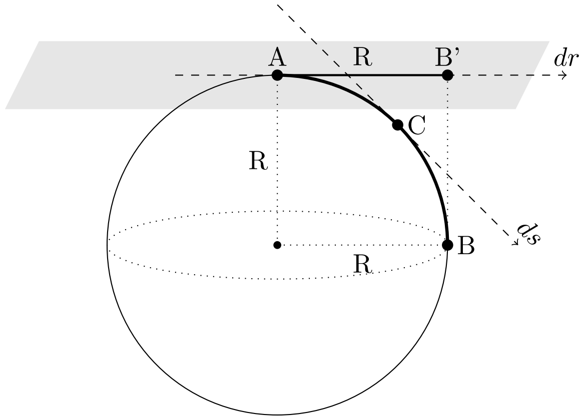

Figure 3: Comparison of flat and curved surfaces.

$$S^2 = \left(\int ds\right)^2 = \int ds^2 = \left(\int_0^R \frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2$$

It can be shown that

$$\int_0^a \frac{dx}{\sqrt{1 - x^2/a^2}} = \frac{\pi}{2}a$$

We thus find S: $S = (\pi/2)a$.

The fact that this metric only holds for half of the spherical surface is also evident from Fig 3: if B was positioned in the bottom hemisphere, then with increasing distance on the spherical shell, the distance on the flat plane would decrease! We can also see the concept that a flat universe is "closed" using Fig.3: for values of $r > R$, we cannot make a connection between the two surfaces any more, a discussion regarding the volume of this space can be seen in §3.

So how can that 2D intelligent creature find out if he is on a curved surface or flat one? eq.7 uses the easy to understand coordinates in his flat surface (r, ϕ), so all he has to do is to measure a certain distance, if that distance is related to dr and $d\phi$ with eq.3, then his universe is flat, if that distance is governed by eq.7 then it's universe is curved in the form of a spherical shell, knowing the distance it can even measure the radius of curvature. In today's cosmology, we still have not been able to reach the sufficient accuracy in our measurements to distinguish between these two equations.

2 Metric of a 3D spherical shell

Please keep the analogy between this section and §1 in mind while reading this. In a flat 3D universe, two of the most common methods to define distance between two points is by using the 3D cartesian coordinates (x, y, z) or (r, θ, ϕ). For the cartesian coordinates distance is written as:

$$ds^2 = dx^2 + dy^2 + dz^2$$

using the spherical coordinates we can write the distance as:

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (9)$$

Like the 2D creature in §1 that could not imagine a 2D surface in a 3D space, we cannot imagine a 3D surface in a 4D space. Such surfaces, that have one dimention less than the space they exist in are called a "hypersurface". So I cannot use figures any more, like that creature in §1, you can only rely on your mathematics, but when ever imagination is needed you can put your self in that creature's shoes and imagine the conditions.

So like §1, we assume an extra dimention, to help us in the mathematics. Let's call it w. So, distance in its most general form can be written as:

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dw^2 \quad (10)$$

Exactly the same mathematics as we applied in §1 can be repeated to get to eq.6, but this time instead of dz, we have found the relation between dw and spherical coordinate r. Exactly similar to §1, we just have to square this and replace it in eq.10, we then take $K = 1/R$ and multiply the whole right side by the scale factor $a(t)$ to get:

$$ds^2 = a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \quad (11)$$

So similar to §1, to find out if our space is curved or not, we simply have to find the distance to a point we know the distance of and see if it changes with $K = 0$ or $K = 1$ in eq.11. Unfortunately this process is much easier said than done!

3 Volume of a Closed Universe

Lets have a look at how the volume of a closed FRW universe is $2\pi^2 a^3(t)x_u^3$.

In an orthogonal space (where $dq_i dq_j = \delta_{ij}$): $dV = \prod_i \sqrt{h_i}dq_i$, where h_i are the diagonal members of the h_{ij} the metric tensor. For example, for the three-dimentional spherical coordinates (eq.9): where $h_r = 1$; $h_\theta = r^2$ and $h_\phi = r^2\sin^2\theta$, $dv = dr \times rd\theta \times r\sin\theta d\phi$. So breaking the FRW metric (eq.11) into its components: $\sqrt{h_x} = a(t)/\sqrt{1 - Kr^2}$, $\sqrt{h_\theta} = a(t)r$, $\sqrt{h_\phi} = a(t)r\sin\theta$. So;

$$V = a^3(t)\int_{r=0}^{\infty}\frac{r^2dr}{\sqrt{1 - Kr^2}}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin\theta d\theta d\phi$$
$$= 4\pi a^3(t)\int_{r=0}^{\infty}\frac{r^2dx}{\sqrt{1 - Kr^2}}$$

If we take $K = 1/R^2$; R being real (closed FRW universe), then the upper limit of the integral cannot extend to $r > R$ and the volume will be: $\pi^2 a^3(t)R^3$. But we have to have in mind that in the derivation of the metric for $K = 1$, only half of the space was considered (positive in the 4th dimention), so the final value has to be multiplied by 2. The plots of $V/a^3(t)$ can be seen in Fig. 4 for all three cases of $K = \{1, 0, -1\}$.

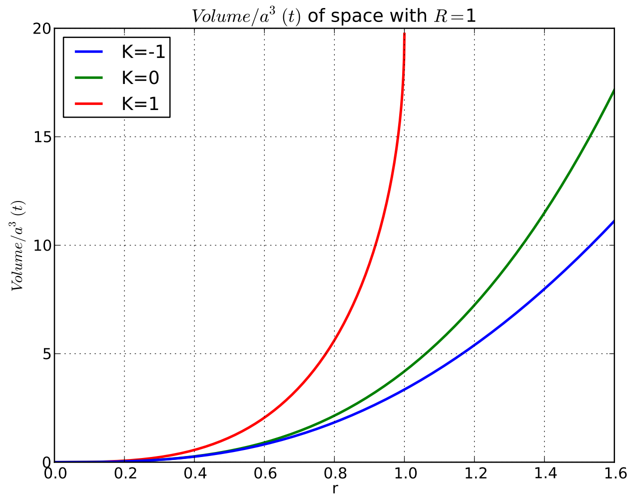

Figure 4: Plots to show how the volume of the universe is "closed" for the k=1 cosmologies but "open" for the k=0 or k=1 (which extend to infinity)

4 Evolution of densities

Another thing that books don't usually elaborate on is the evolution of densities. I will assume you know how we reached:

$$H(z) = H_0 E(z)$$

where:

$$E(z) = [\Omega_{\Lambda,0} + \Omega_K(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{\gamma,0}(1+z)^4]$$

where we take the values of the current Cosmic density parameters from observational results. $\Omega_{\Lambda,0} = 0.75 \pm 0.2$, $\Omega_K \approx 0$, $\Omega_{m,0} = 0.27 \pm 0.05$, $\Omega_{\gamma,0} = 4.2 \times 10^{-5} h^{-2}$ where $h = H_0/100$ and $H_0 \approx 71$. Using this definition for $E(z)$ we can find the evolution of the different cosmic density parameters. We first look at the definition of the ciritical density:

$$\rho_{crit}(z) = \frac{3H^2(z)}{8\pi G} = \frac{3H_0^2(z)}{8\pi G} E^2(z) = \rho_{crit,0} E^2(z)$$

Before continuing lets have a look at the plots for this eqation in Fig.5, where you can see the evolution of the critical density both in terms of redshift and the age of the universe. I have taken $\rho_{crit,0} = 1$ in this figure. I have derived the age of the universe at each redshift based on this integral:

$$t(z) = \int_0^{a(z)} \frac{da}{\dot{a}} = \frac{1}{H_0} \int_z^{\infty} \frac{dz}{(1+z)E(z)} \tag{12}$$

Lets have a closer look at Fig.5; As you can see, both the vertical and horizontal axis' have a logarithmic scale. The $z = 0$ in the top figure corresponds to the local universe while $t = 0$ in the bottom corresponds to the beginning of the universe, so in the redshift (top) plot, the local universe can be seen more precisely while in the universe age (bottom) plot, the behavior in the early universe is more clearly seen. This is a point that although mathematically obvious, but is rarely elaborated on in Cosmology books: that the critical density it's self evolves and has become significantly smaller compared to the early universe, this is logical since the Energy of the universe has not changed but it's volume has! You have to keep this in mind for the following:

Looking at the definition of the cosmic density parameters in general:

$$\Omega_x = \frac{\rho_x}{\rho_{crit}} = \frac{\rho_{x,0}(1+z)^{3(1+w)}}{\rho_{crit,0}E^2(z)} = \Omega_{x,0}\frac{(1+z)^{3(1+w)}}{E^2(z)} \tag{13}$$

Where $w = 0$ for Matter(Ω_m), $1/3$ for relativistic particles(Ω_γ), -1 for Dark energy(Ω_Λ) and $-1/3$ for Curvature(Ω_K). From eq.13 we can find the evolution of each one of these components: Fig.6. Just like above, the top (Redshift) plot has more emphasis on the near by ($z < 1$) universe while the bottom

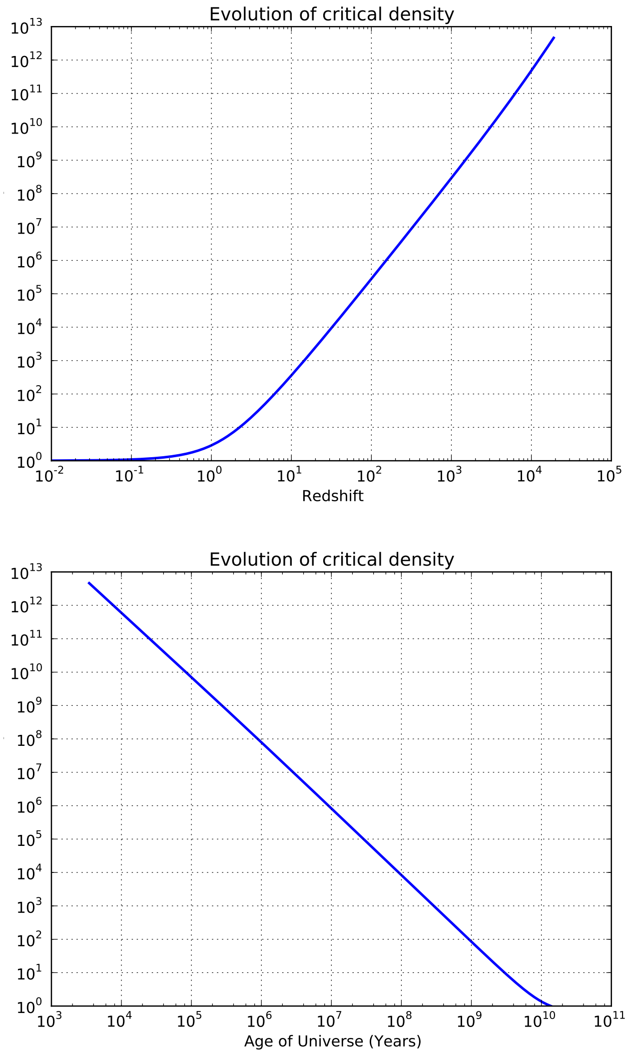

Figure 5: Evolution of Critical Density as a function of redshift (top) and Age of Universe (bottom)

(Age of Universe) plot emphasizes more on the very early ages.

You can see the points of Matter-Radiation equality and Matter-Dark Energy Equality in Fig.6 as the colliding points of the two respective curves, I have put thin black lines to distinguish these points. Also, to put things into a larger context, I have also added the redshift and age of the universe at the time of decoupling; When the Photons and Matter did not significantly interact with each other any more and thus the Cosmic Microwave Background originated. In the figures I have distinguished this point with a thicker gray vertical line. You can see the appropriate values here.

$$z_{m,r} \approx 3149, \qquad t_{m,r} \approx 5.9 \times 10^4 years$$

$$z_{dec} \approx 1100, \qquad t_{dec} \approx 3.7 \times 10^5 years$$

$$z_{m,\Lambda} \approx 0.39, \qquad t_{m,\Lambda} \approx 9.6 \times 10^9 years$$

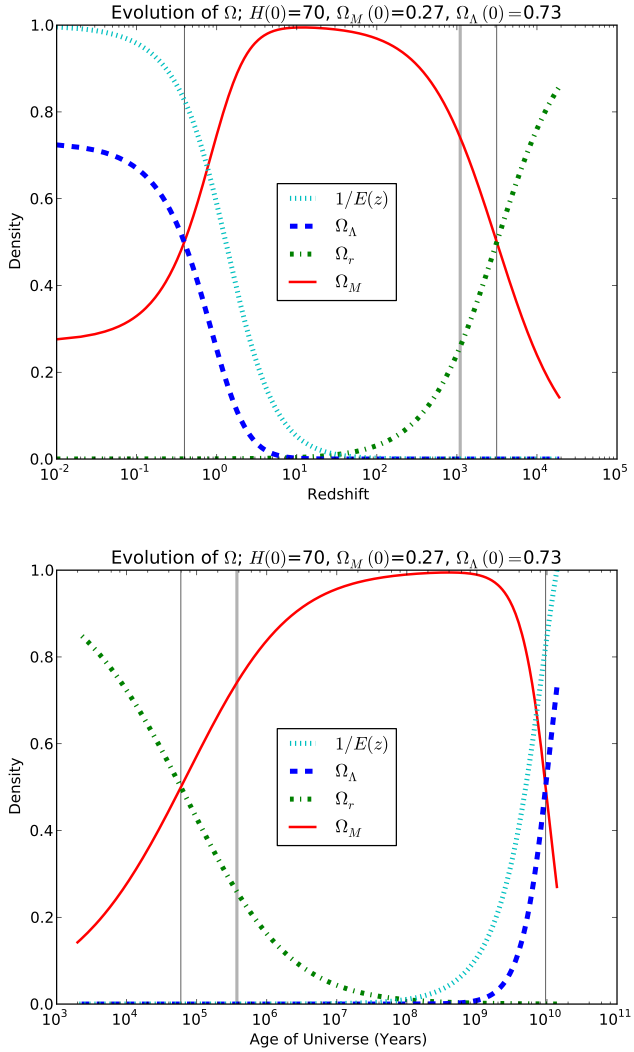

Figure 6: Evolution of the three components of energy density in the Universe as a function of redshift (top) and Age of Universe (bottom). $1/E(z)$ is also included in these plots since it is a fundamental part of each and for comparison. The two thin horizontal black lines are at $z_{m,r} = 3149$ and $z_{m,\Lambda} = 0.39$ in the top figure and $t_{m,r} = 5.9 \times 10^4 years$ and $t_{m,\Lambda} = 9.6 \times 10^9 years$ for the bottom picture. They show the redshifts and Age of universe in the two epochs of density equality.