Chap.1 Kinematics and Dynamics of Galactic Stars

- Orbits of stars
  - Orbits in spherical and tri-axial potentials
  - Orbits in Stäckel potentials, Action integrals
- Kinematics of stars
  - (U,V,W), LSR, solar motion, Galactic constants
- Distribution functions of stars
  - Schwarzschild, modeling distribution functions
- Jeans equations
  - Jeans theorem, spherical system, asymmetric drift
- Virial theorem

1. Orbits of stars

Galaxy structure: superposition of stellar orbits

- Orbits in a spherical potential Φ(r)

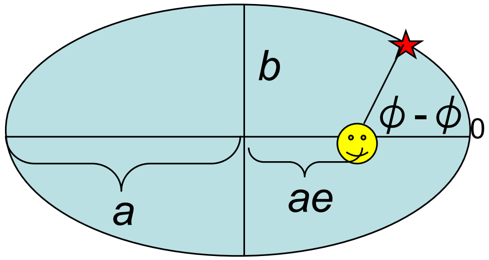
  L*=*r*×*dr*/*dt =const.

  ⇒ confined to the 2D orbital plane ⇒ coordinates (r, ϕ)

E.g. Kepler potential (by a point mass: M) $\Phi(r) = -GM/r$

$$\begin{cases} L = r^2 d\phi/dt \\ E = \frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2r^2} - \frac{GM}{r} \end{cases}$$

⇨ $$\frac{a(1-e^2)}{r} = 1 + e\cos(\phi - \phi_0)$$ equation for an ellipse (orbit)



$e \equiv \sqrt{1 - 2|E|L^2/(G^2M^2)}$ eccentricity

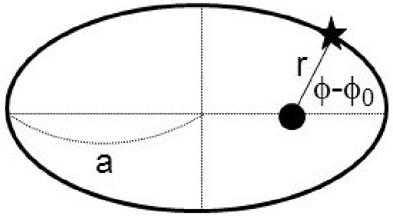
$a \equiv L^2/GM(1-e^2)$ semi-major axis

$r_{\rm pr} = a(1-e)$ pericenter

$r_{\rm ap} = a(1+e)$ apocenter

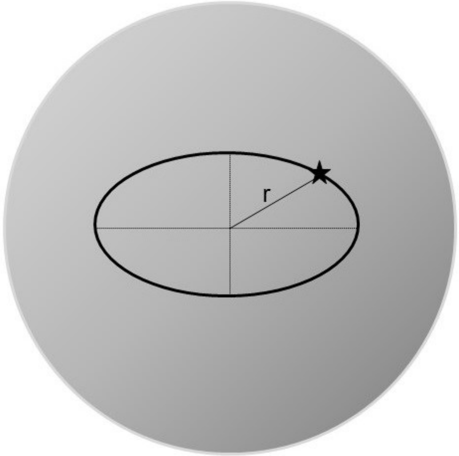
⇨ $$e = \frac{r_{\rm ap} - r_{\rm pr}}{r_{\rm ap} + r_{\rm pr}}$$

Kepler motion
(orbit in a point-mass potential)



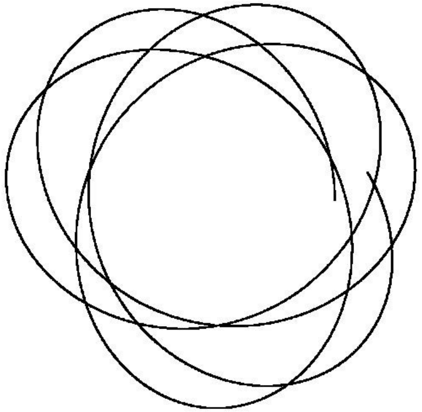
$\Delta\phi = 2\pi$ over one period of radial oscillation
⇒ closed orbit

Orbit inside a uniform sphere



$M(<r) = 4\pi\rho r^3/3$
$F(r) = -GM(<r)/r^2$
$\propto -r$
$F_x \propto -x$, $F_y \propto -y$
Simple oscillator

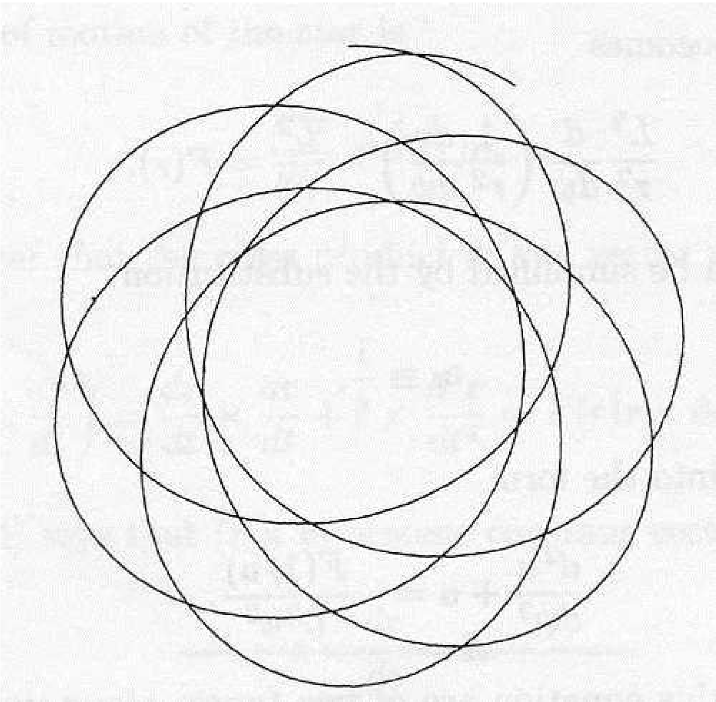
$\Delta\phi = \pi$
closed orbit



Orbit in a gravitational potential provided by a general spherical mass distribution

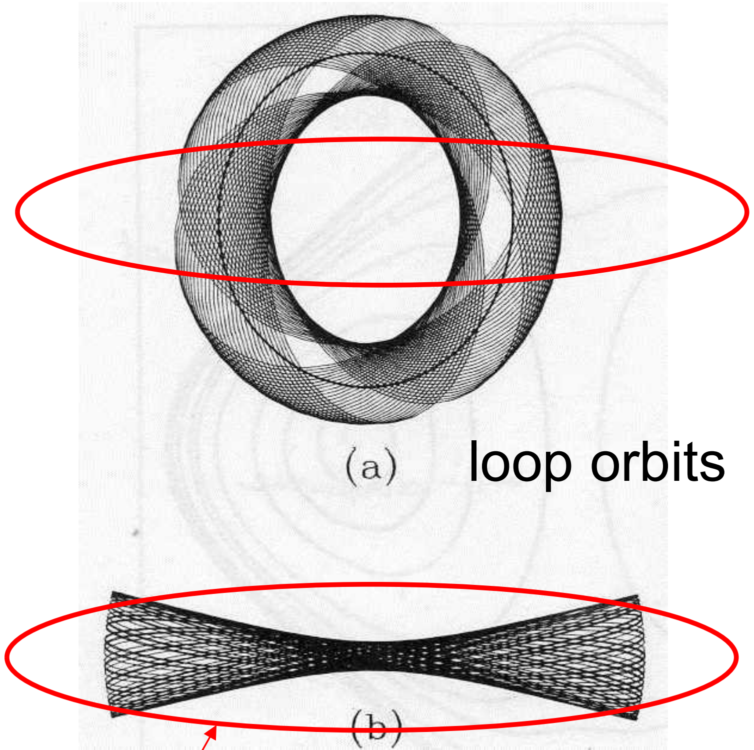
$\pi < \Delta\phi < 2\pi$
Rosette orbit
(non-closed)

・2D axisymmetric Φ(R)



loop orbits

・2D non-axisymmetric Φ(x,y)
(Nonrotating bar potential)



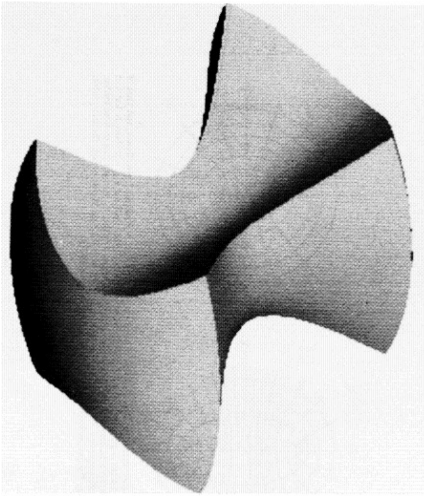
loop orbits

box orbits

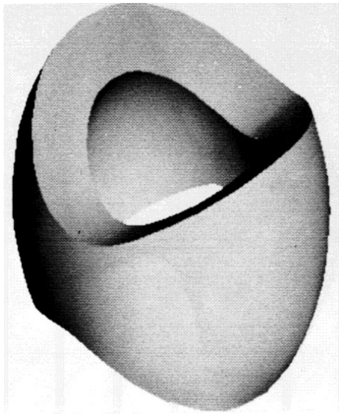
Orientation of a bar

Orbits in a (nonrotating) triaxial potential

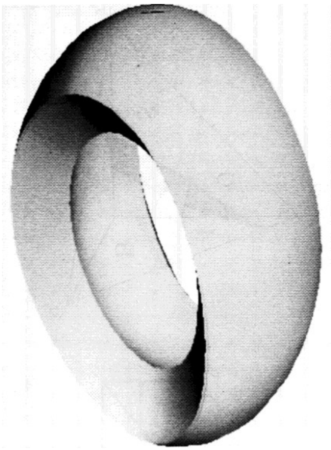
Box orbit



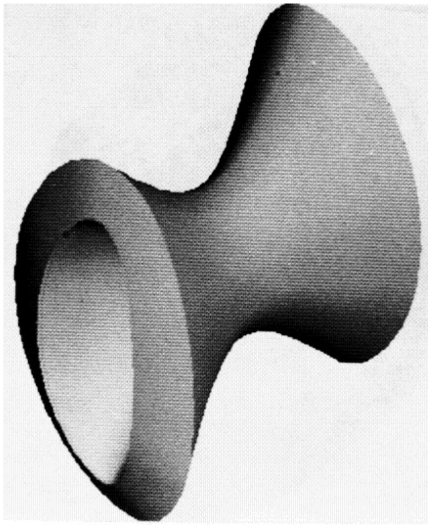
Short-axis tube orbit



Outer long-axis
tube orbit



Inner long-axis
tube orbit



Statler 1987, ApJ, 321, 113

- Stäckel potential
  - Hamilton-Jacobi eq. is separable ⇒ eq. of motions is solvable independently in each spatial coordinate
  - Integral of motion $I_i(\boldsymbol{x},\boldsymbol{v})$ i=1,3

  Only <u>regular orbits</u> exist (de Zeeuw 1985, MNRAS, 216, 273)

  explicit expressions for $I_1(\boldsymbol{x},\boldsymbol{v})$ (=E), $I_2(\boldsymbol{x},\boldsymbol{v})$, $I_3(\boldsymbol{x},\boldsymbol{v})$

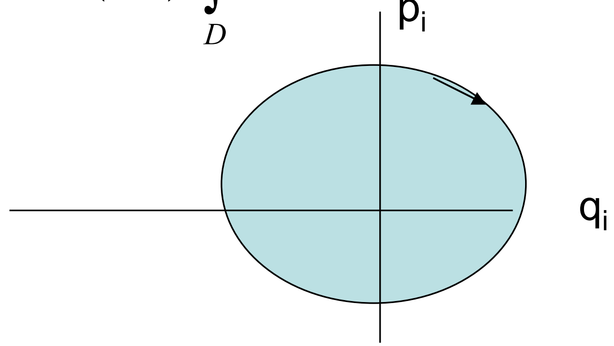
- Action integrals

  $J_i(E,I_2,I_3)$ i=1,3

$$J_i = \frac{1}{2\pi}\oint p_i dq_i$$

  - Adiabatic invariance
  - $J_i \geq 0$ for bound orbits
  - Phase volume

$$V = \int_D d^3x d^3v = \int_D d^3J d^3\theta = (2\pi)^3 \int_D d^3J$$



Canonical transformation $(\boldsymbol{q},\boldsymbol{p}) \rightarrow (\boldsymbol{\theta},\boldsymbol{J})$

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i} = \omega_i, \dot{J}_i = -\frac{\partial H}{\partial \theta_i} = 0$$

Action integrals for a Kepler motion

$$J_r = \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} p_r dr = \frac{\sqrt{2}}{\pi}\int_{r_{\min}}^{r_{\max}} \sqrt{E - \frac{L^2}{2r^2} + \frac{GM}{r}} dr = \frac{GM}{\sqrt{2|E|}} - L = L\left[\frac{1}{\sqrt{1-e^2}} - 1\right]$$

$$J_\theta = \frac{1}{\pi}\int_{\theta_{\min}}^{\theta_{\max}} \sqrt{L^2 - \frac{{p_\phi}^2}{\sin^2\theta}} d\theta = L - |J_\phi|$$

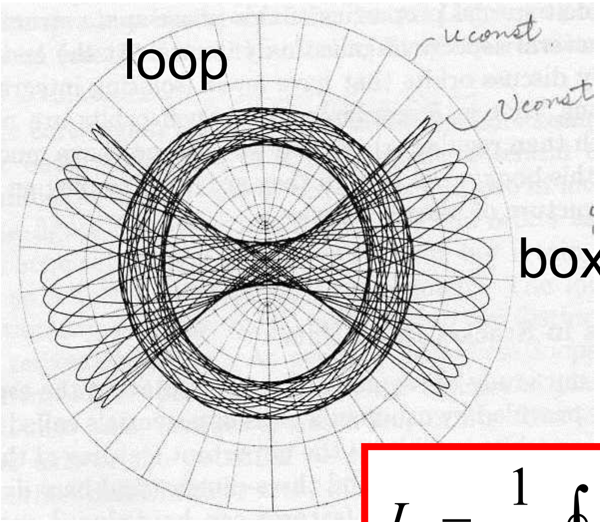
$$J_\phi = \frac{1}{2\pi}\oint p_\phi d\phi = p_\phi$$

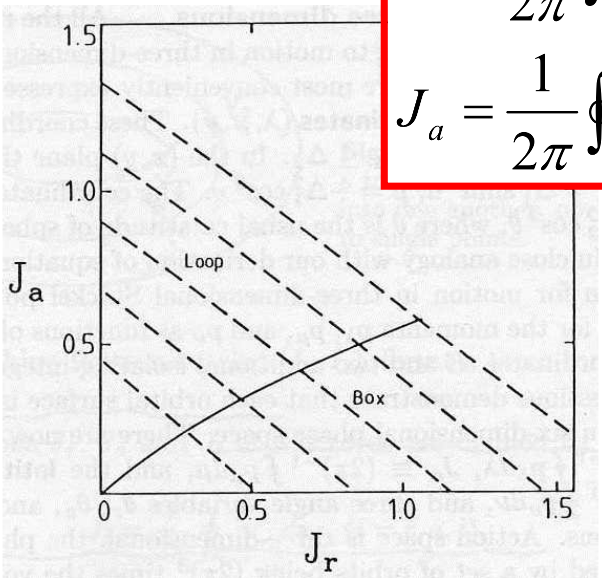
$L=|J_\phi|+J_\theta$: adiabatic invariance

⇒ *Orbital eccentricity: e* is an adiabatic invariance as well

(a conserved quantity when the change of a gravitational potential is sufficiently slow compared to its dynamical time scale)

- 2D non-axisymmetric $\Phi(x,y) \Rightarrow \Phi(u,v)$

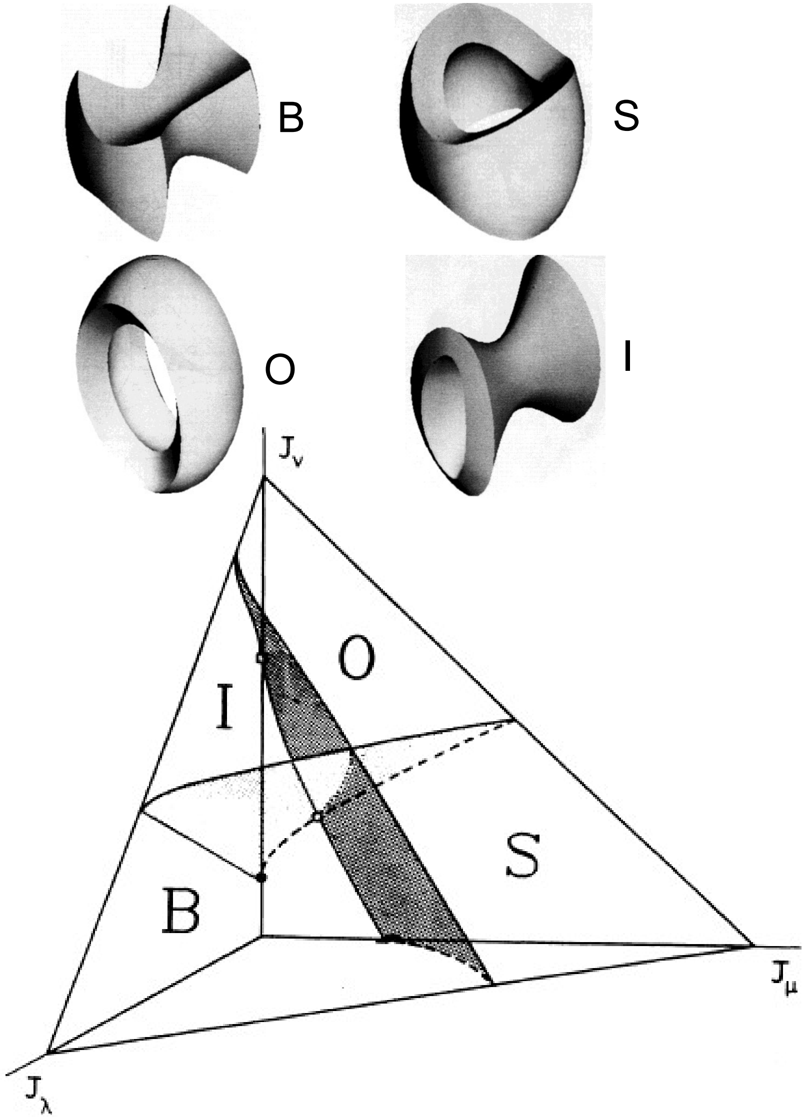




$$J_r = \frac{1}{2\pi} \oint p_u du$$

$$J_a = \frac{1}{2\pi} \oint p_v dv$$

- 3D triaxial $\Phi(\lambda,\mu,\nu)$



2. Kinematics of stars

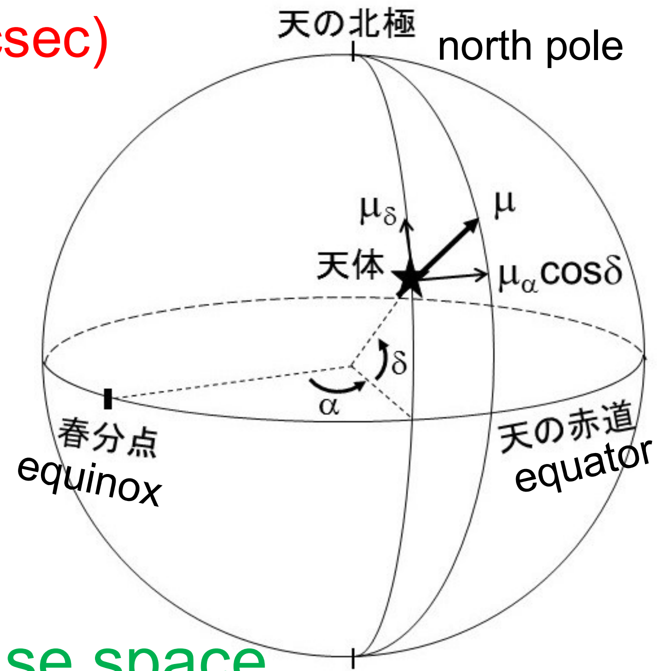
Description of stellar kinematics observed from the Sun

- Observed kinematical quantities
  - Line of sight velocity: V_{rad},
  - Distance: D (pc) = 1 / π (arcsec)

    or D (kpc) = 1 / π (mas, milli-arcsec)
  - Proper motion: $\mu = [(\mu_\alpha \cos\delta)^2 + (\mu_\delta)^2]^{1/2}$

    [unit: arcsec (″) /yr

    or mas (milli-arcsec: 10^{-3}″) /yr]



Tangential velocity:

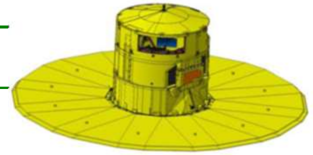
V_{tan}(km/s)=4.74 D(pc) μ(arcsec/yr)

= 4.74 D(kpc) μ(mas/yr)

– (α,δ), D, V_{rad}, (μ_α,μ_δ)

→3d position＋3d velocity→6d phase space

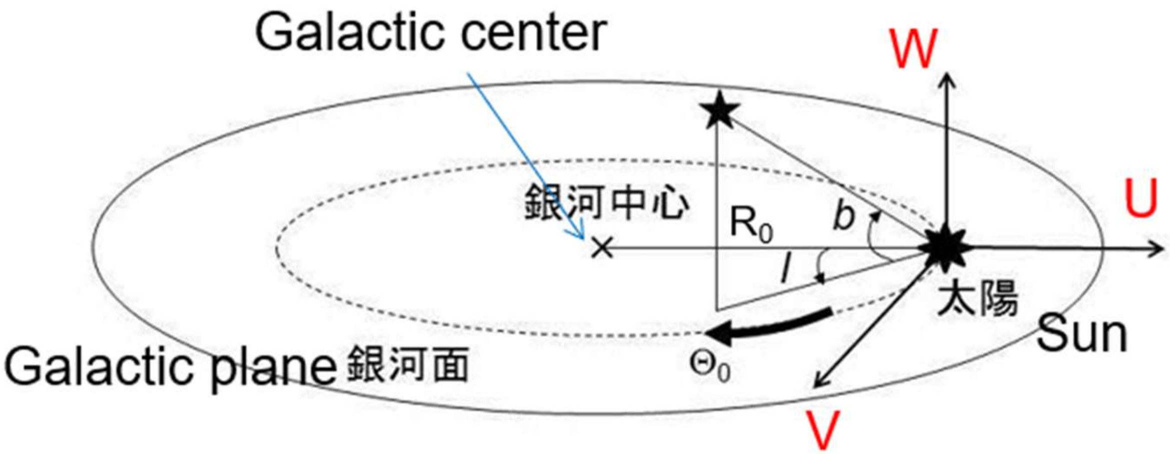
Astrometry Satellites: Hipparcos & Gaia



	1989~1993 Hipparcos	2013~2021 Gaia
Magnitude limit	12 mag	20 mag
Completeness	7.3 – 9.0 mag	20 mag
Bright limit	0 mag	6 mag
Number of objects	120,000	26 million to V = 15 250 million to V = 18 1000 million to V = 20
Effective distance	1 kpc	50 kpc
Quasars	1 (3C 273)	500,000
Galaxies	None	1,000,000
Accuracy	1 milliarcsec	7 μarcsec at V = 10 10 – 25 μarcsec at V = 15 300 μarcsec at V = 20
Photometry	2-colour (B and V)	Low-res. spectra to V = 20
Radial velocity	None	15 km s^{-1} to V = 17
Observing	Pre-selected	Complete and unbiased

Gaia: 10μas = 10% error @distance 10kpc, 10μas/yr = 1km/s @20kpc
Hipparcos: 1mas = 10% error @distance 100pc, 1mas/yr = 5km/s @ 1kpc

The Local Standard of Rest (LSR) & UVW velocities



- The LSR: a circular orbit at ($R=R_0$, $V_{rot} = \Theta_0$)
- (U,V,W): star's velocities relative to the LSR
- (U_{sun},V_{sun},W_{sun}): solar motion relative to the LSR
- Observed star's velocities from the Sun (heliocentric)

  $(U-U_{sun}, V-V_{sun}, W-W_{sun}) \simeq (V_R, V_\Phi - \Theta_0, V_z)$ in cylindrical coords.

Galactic constants: Θ_0, R_0, (U_{sun},V_{sun},W_{sun})

Determination of Galactic Constants

LSR
(Θ_0, R_0)

- Rotational velocity of the LSR: Θ_0
  - Oort constants (A,B) → Ω_0=A-B → R_0 given → $\Theta_0=R_0$ (A-B)
  - Motion relative to Pop II system, but $\langle\Theta\rangle=0$ is assumed
  - Proper motion of Sgr A* → R_0 → Θ_0

    if Sgr A* is fixed at the center and the LSR has Θ_0=220km/s, then $\Theta_0=4.74D\mu_\ell$ → proper motion along Galactic long.: $\mu_\ell \sim 5.8$ mas/yr
- Solar position: R_0
  - The center of halo tracers (GCs, RR Lyr, Mira variables in the bulge)
  - Parallax of Sgr A*: p(mas) = $(D/\text{kpc})^{-1}$ = 0.1 mas
  - Stellar motions near Sgr A* ("binary method") Salim & Gould 1999

➢ Kerr & Lynden-Bell (1986, MN, 221, 1023)

Θ_0=220 km/s, R_0=8.5 kpc (IAU standards)

➢ Recent trend: Θ_0 > 220 km/s, R_0 ~ 8 kpc

Solar motion
(U_{sun}, V_{sun}, W_{sun})

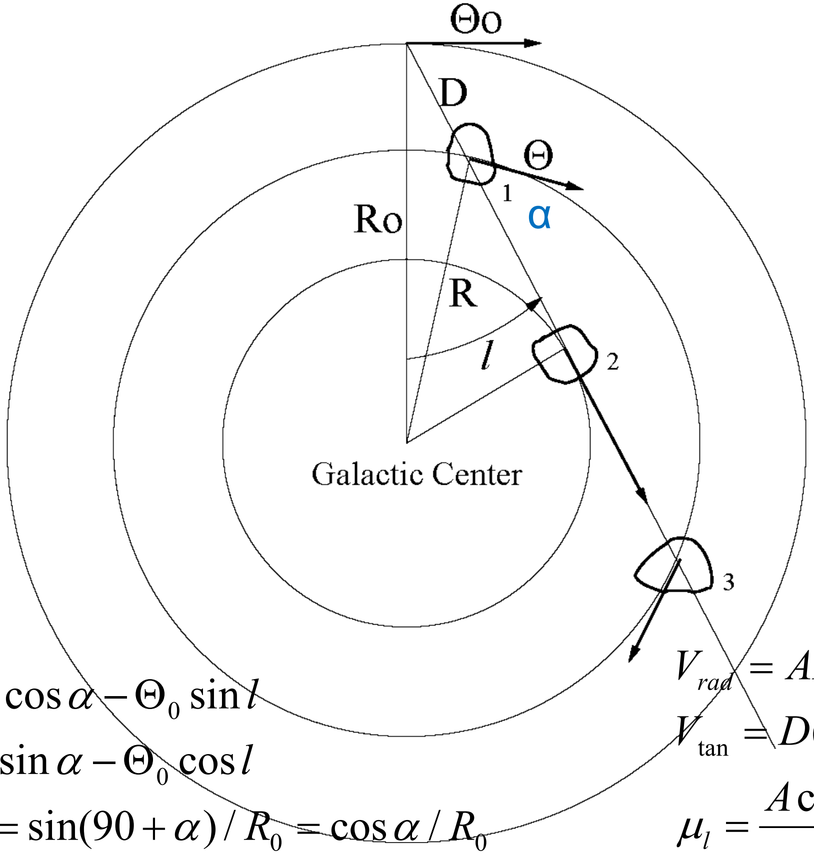
* Delhaye 1965 using A stars, K giants, M dwarfs

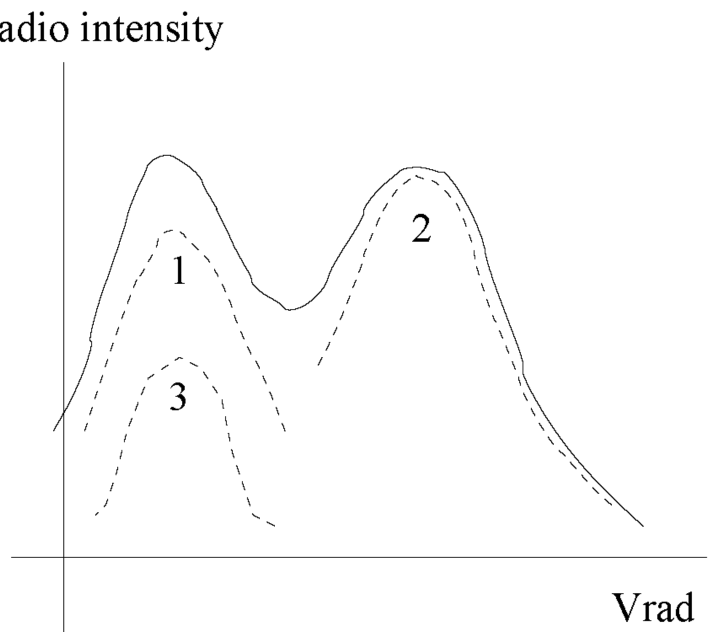
(U_{sun}, V_{sun}, W_{sun}) = (-9, 12, 7) km/s, (l,b)=(53,25)

* More recent result (Schönrich +10, Coşkunoğglu+11)

(U_{sun}, V_{sun}, W_{sun}) = (-11.10, <u>12.24</u>, 7.25) km/s

Determination of the rotation curve





$$V_{rad} = \Theta \cos\alpha - \Theta_0 \sin l$$

$$V_{\tan} = \Theta \sin\alpha - \Theta_0 \cos l$$

$$\sin l / R = \sin(90+\alpha)/R_0 = \cos\alpha / R_0$$

$$R\cos(90-\alpha) = R\sin\alpha = R_0 \cos l - D$$

$$V_{rad} = (\Omega - \Omega_0) R_0 \sin l$$

$$V_{\tan} = (\Omega - \Omega_0) R_0 \cos l - \Omega D$$

$\Rightarrow$ $(R_0\text{-}R)/R_0 << 1$ $\Rightarrow$

$$V_{rad} = AD\sin 2l$$

$$V_{\tan} = D(A\cos 2l + B)$$

$$\mu_l = \frac{A\cos 2l + B}{4.74}$$

$$A \equiv \frac{1}{2}\left[\frac{\Theta_0}{R_0} - \left(\frac{d\Theta}{dR}\right)_{R_0}\right]$$

$$B \equiv -\frac{1}{2}\left[\frac{\Theta_0}{R_0} + \left(\frac{d\Theta}{dR}\right)_{R_0}\right]$$

A, B: Oort Constants

$$\Omega_0 = \Theta_0 / R_0 = A - B$$

$$\left(\frac{d\Theta}{dR}\right)_{R_0} = -(A+B)$$

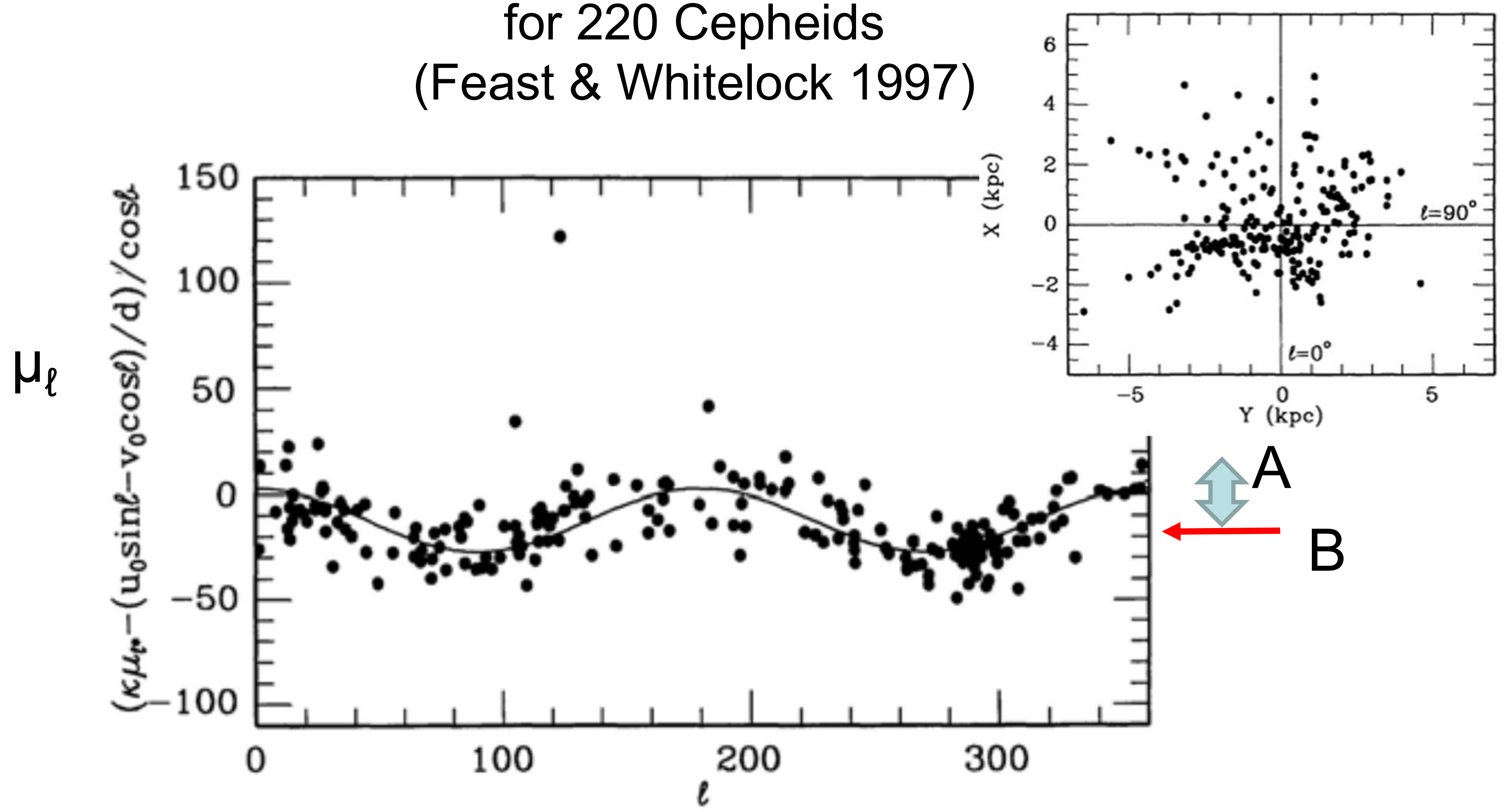
V_{rad} is max at $R = R_0 \sin l \Rightarrow \Theta(R)$

$$\frac{\partial V_{rad}}{\partial D} = 0 \Rightarrow \partial R/\partial D = 0$$

$$R^2 = D^2 + R_0^2 - 2DR_0 \cos l$$

$$\Rightarrow D = R_0 \cos l \quad \Rightarrow R = R_0 \sin l$$

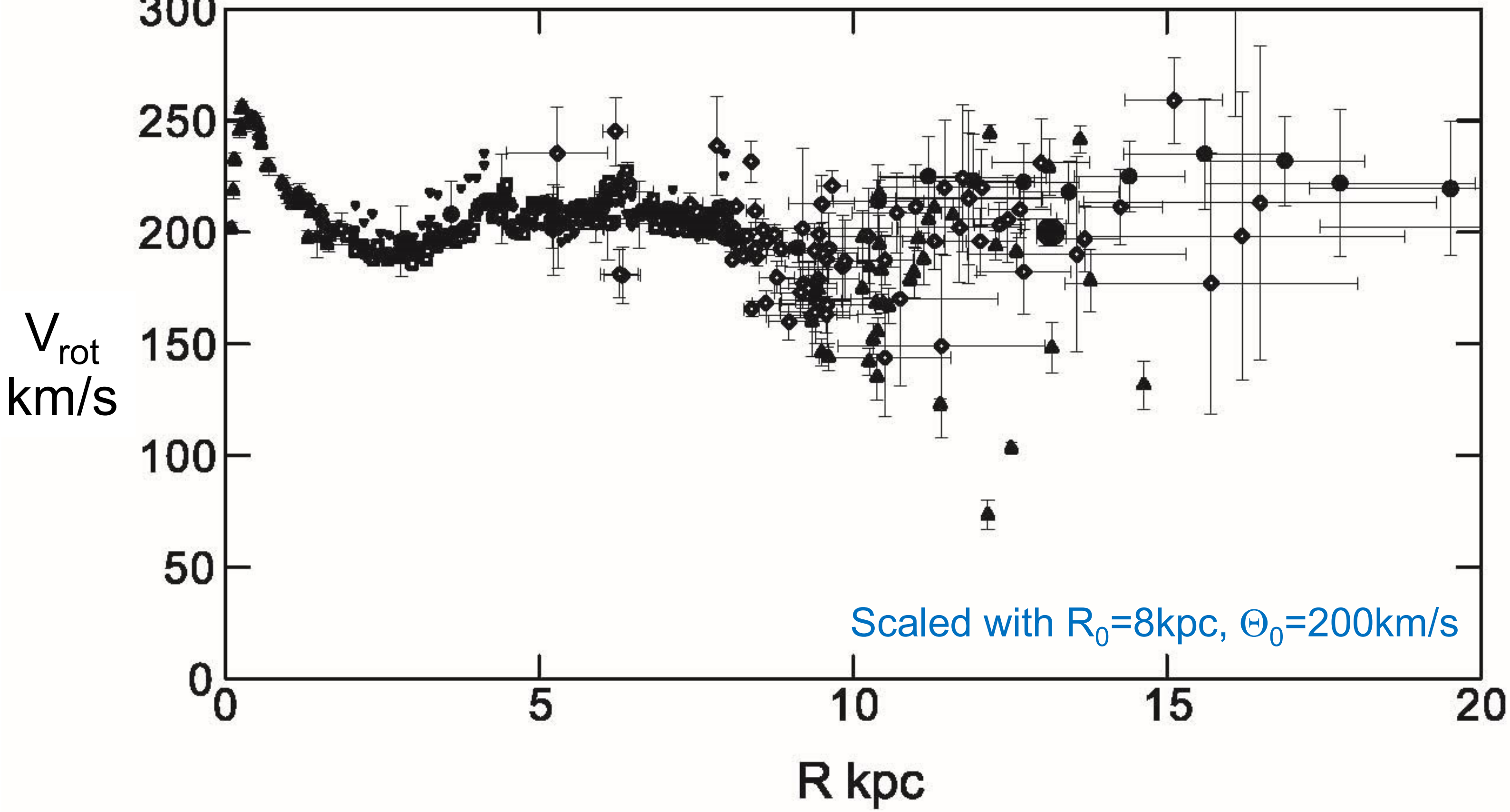
Hipparcos proper motions toward Galactic longitude for 220 Cepheids (Feast & Whitelock 1997)



$\mu_\ell \propto V_{tan}/D = A\cos 2\ell + B \Rightarrow A, B$

Rotation curve of the Milky Way

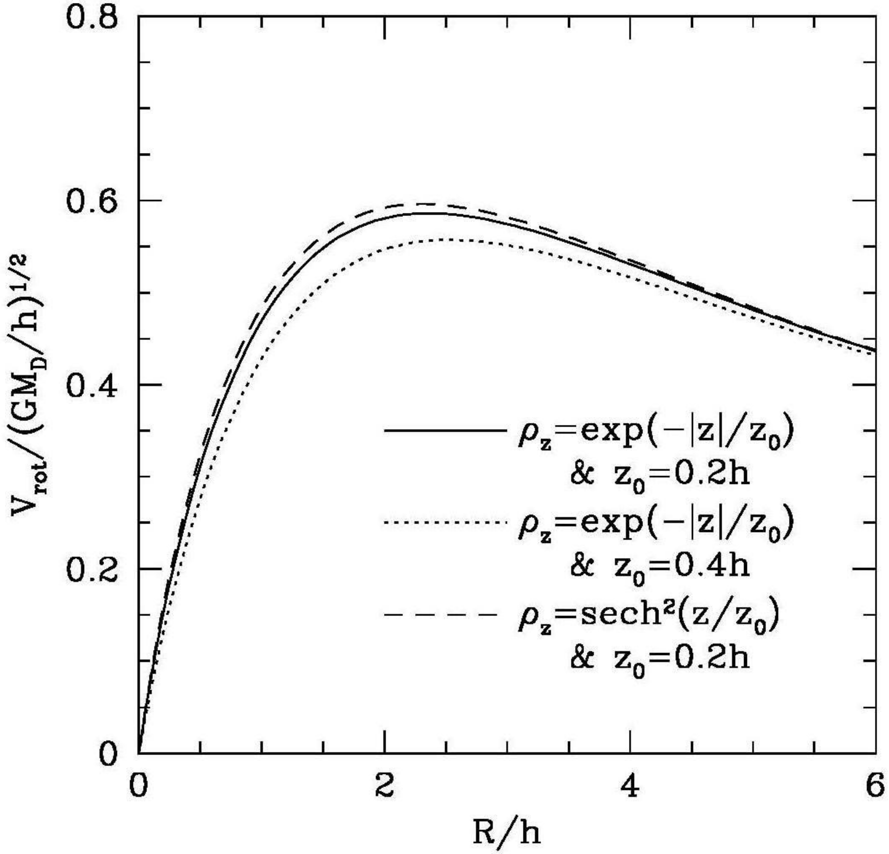
Sofue et al. 2009, PASJ, 61, 227

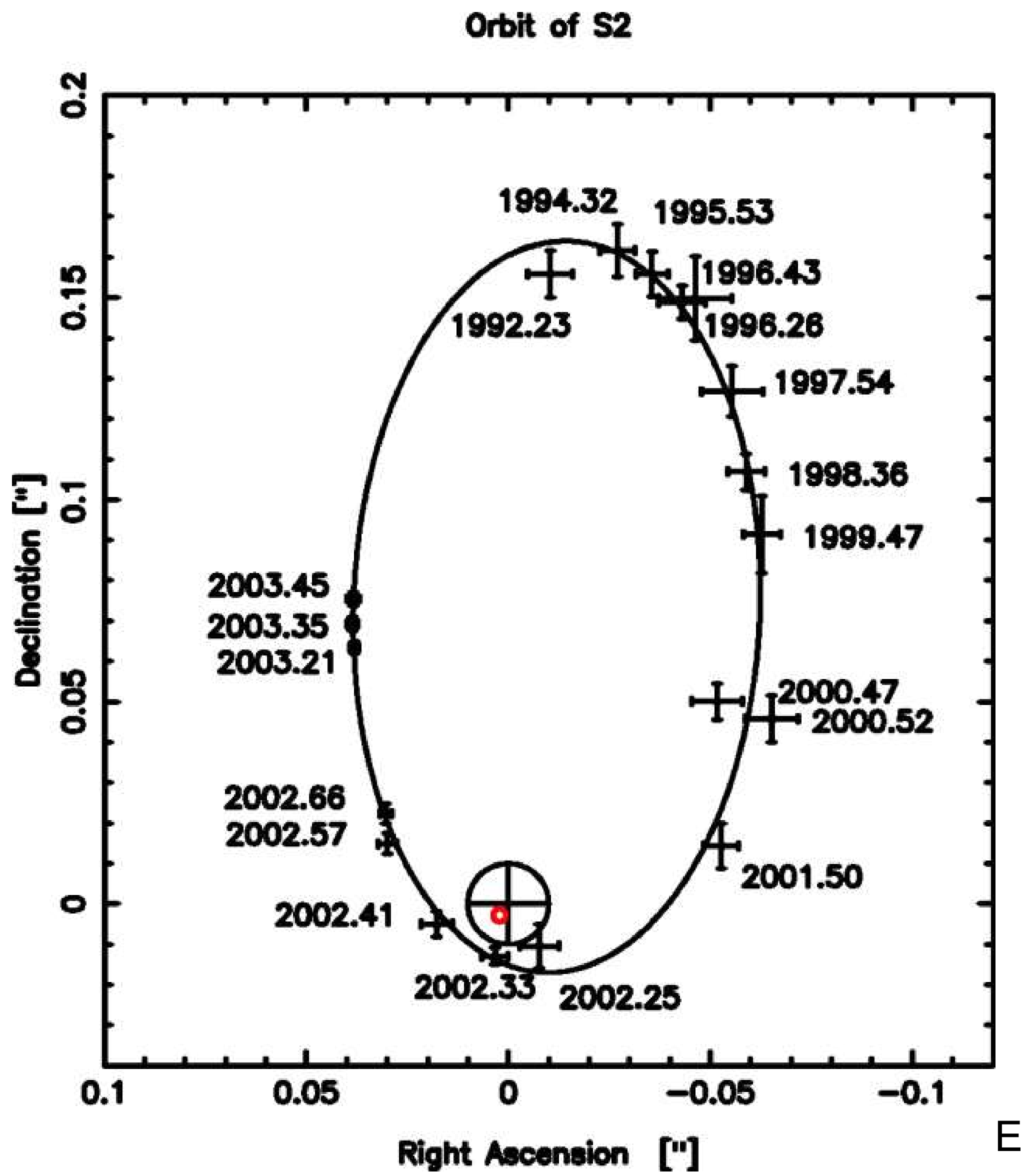


See also, Gunn, Knapp, Tremaine 1979, AJ, 84, 1181;
Fich & Tremaine 1991, ARAA, 29, 409

Effect of disk thickness on rotation curve at the disk plane

V_{rot} provided by an exponential disk: $\rho(R,z) = \rho_0 \exp(-R/h)\rho_z(z)$





Geometric determination of the distance to the Galactic Center

orbital eclipse vs. angular sep.
→ R_0

$R_0 = 7.94 \pm 0.42$kpc

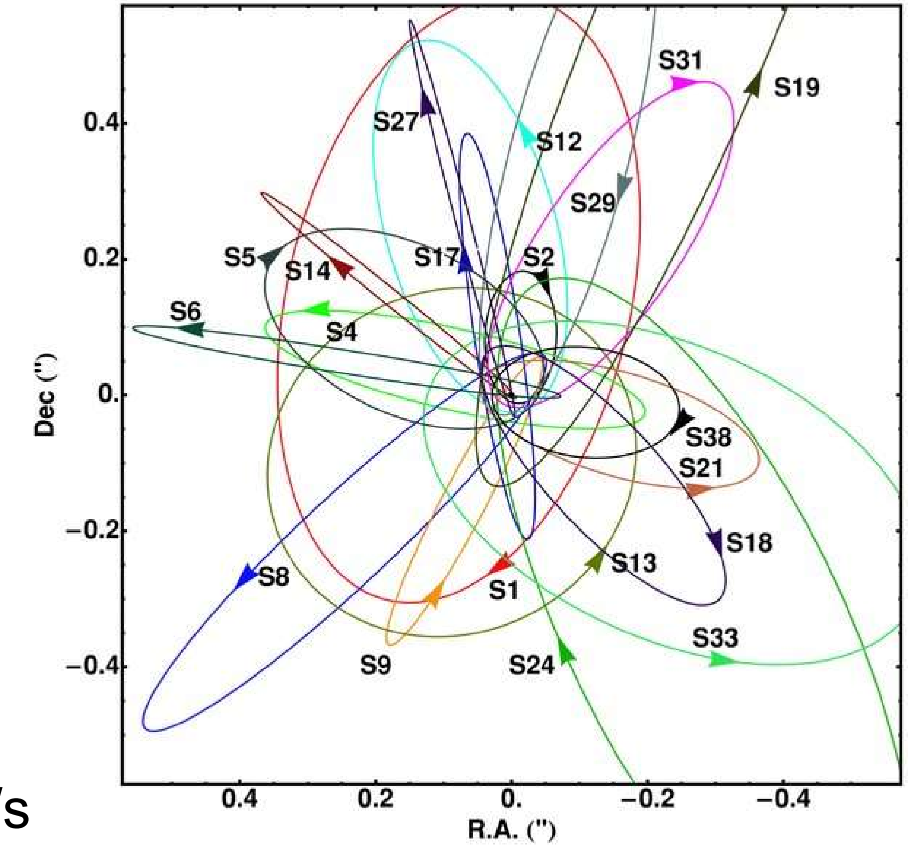
Eisenhauer et al. 2003, ApJ, 597, L121

Using more orbits

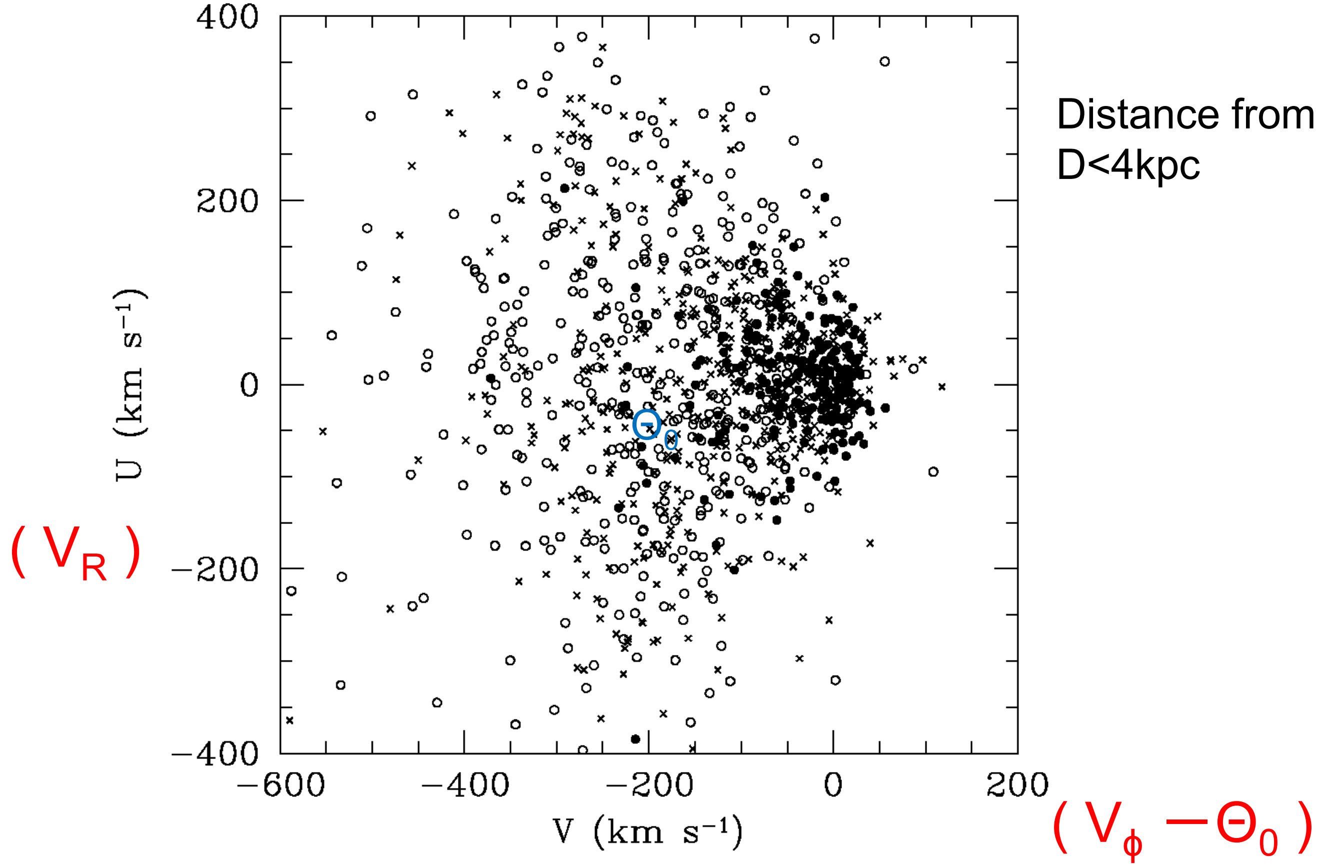
Gillessen et al. 2009:
16 years of monitoring
the orbits of 28 stars
R_0 = 8.33±0.35 kpc

Reid & Brunthaler 2004:
μ_ℓ(SgrA*)=6.379±0.026mas/yr

⇒$(\Theta_0+V_{sun})/R_0$=30.24 km/s/kpc
Then if R_0=8.3 kpc & V_{sun}=12.24 km/s
⇒Θ_0=239 km/s



(U,V) velocities for nearby stars (V_R , $V_\phi - \Theta_0$)

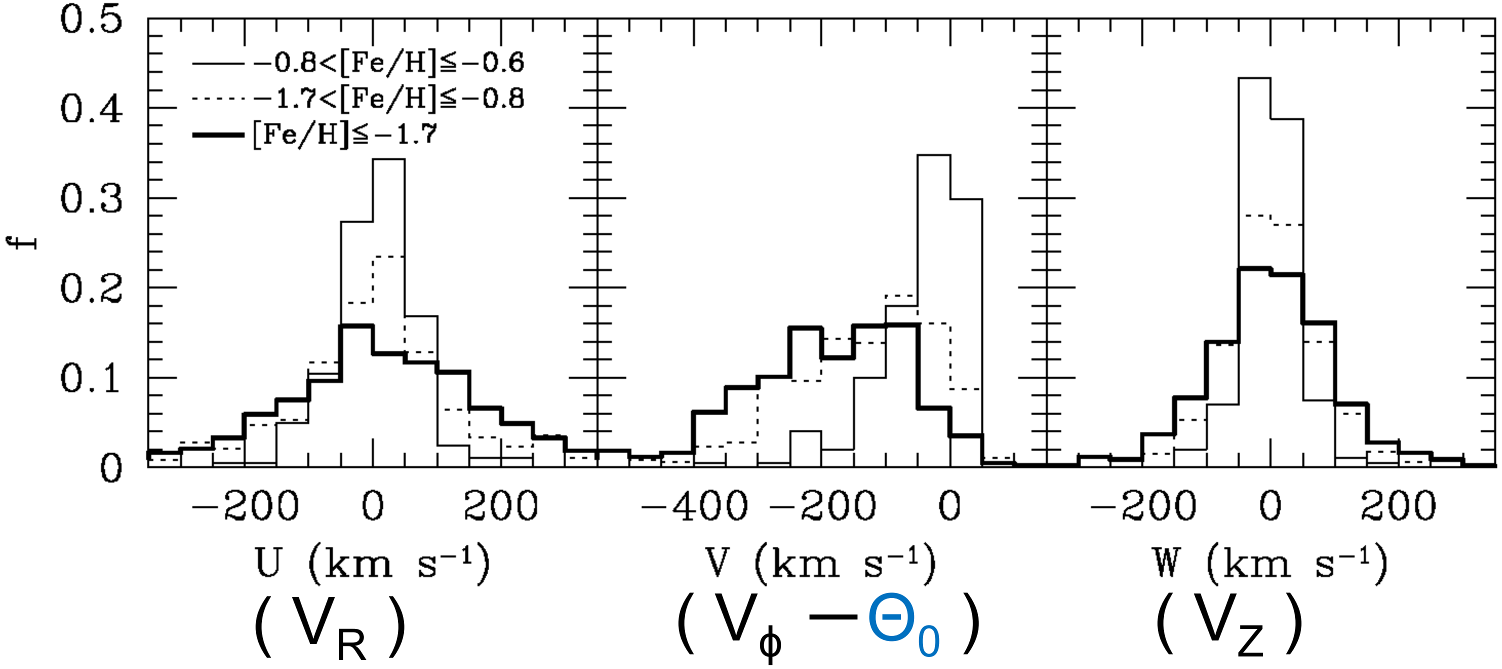


Distance from the Sun
D<4kpc

● -0.8<[Fe/H]≤-0.6

× -1.7<[Fe/H]≤-0.8

○ [Fe/H]≤-1.7



	σ_U	σ_V	σ_W	<V>
[Fe/H]≤-1.7	150 km/s	110 km/s	100 km/s	-200 km/s
-0.8<[Fe/H]≤-0.6	60 km/s	60 km/s	40 km/s	-30 km/s

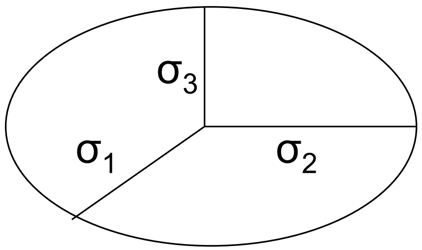
3. Distribution function of stars

- Schwarzschild (1907) model

$$f(v_1, v_2, v_3)dv_1dv_2dv_3 = \frac{dv_1dv_2dv_3}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\exp\left[-\left(\frac{v_1^2}{2\sigma_1^2}+\frac{v_2^2}{2\sigma_2^2}+\frac{v_3^2}{2\sigma_3^2}\right)\right]$$

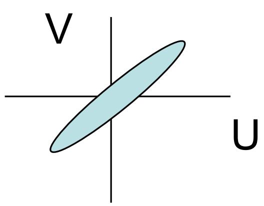
$\sigma_i^2 = \langle(v_i - \langle v_i\rangle)^2\rangle$ velocity dispersion

Velocity ellipsoid



Vertex deviation

σ_i axis does not necessary match the direction of (U,V,W)



Modeling distribution functions

$f(\boldsymbol{r}, \boldsymbol{v}, t) d^3\boldsymbol{r} d^3\boldsymbol{v}$ $(\boldsymbol{r}, \boldsymbol{v})$ phase space

$E(\boldsymbol{r}, \boldsymbol{v})$ $I_2(\boldsymbol{r}, \boldsymbol{v})$ $I_3(\boldsymbol{r}, \boldsymbol{v})$ Integrals of motions

$$n(\boldsymbol{r}) = \int f d^3\boldsymbol{v} \qquad \langle v_i \rangle = \frac{1}{n} \int v_i f d^3\boldsymbol{v}$$

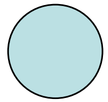
$$\sigma_i^2 = \langle (v_i - \langle v_i \rangle)^2 \rangle \qquad \sigma_{ij}^2 = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$$

Integrals of motions, $I(\boldsymbol{r}, \boldsymbol{v})$, are the solutions to the steady-state collisionless Boltzmann equation

$$\frac{dI}{dt} = \boldsymbol{v} \cdot \nabla I - \nabla \Phi \cdot \frac{\partial I}{\partial \boldsymbol{v}} = 0$$

⇒ Jeans Theorem: $f(E, I_2, I_3)$, $f(J_1, J_2, J_3)$

Some simple cases

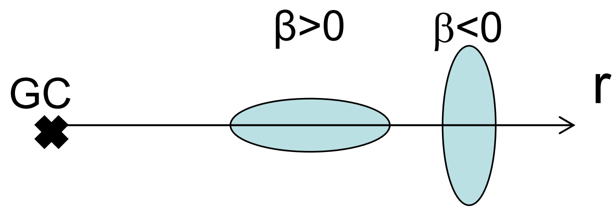


- $f(E)$ isotropic velocity distribution
- $f(E,L_z)$ $L_z=Rv_\phi$ in axisymmetric $\Phi(R,z)$
  - $\sigma_R^2=\sigma_z^2$ (but $\neq\sigma_\phi^2$) anisotropic
  - but $\sigma_R^2\neq\sigma_z^2$ near the Sun $\rightarrow$ presence of I_3

  $(\sigma_U,\sigma_V,\sigma_W) \approx (150,110,100)$ km/s for halo stars
- $f(E,L)$ L: total angular momentum
  - $v_r=v\cos\eta$, $v_\theta=v\sin\eta\cos\psi$, $v_\phi=v\sin\eta\sin\psi$
  - $v_t^2=v_\theta^2+v_\phi^2=v^2\sin^2\eta$, $L=|rv_t|=|rv\sin\eta|$
  - $\sigma_\theta^2=\sigma_\phi^2\neq\sigma_r^2$ anisotropic
  - $\beta(r)=1-\sigma_\theta^2/\sigma_r^2$, $\beta\leq1$

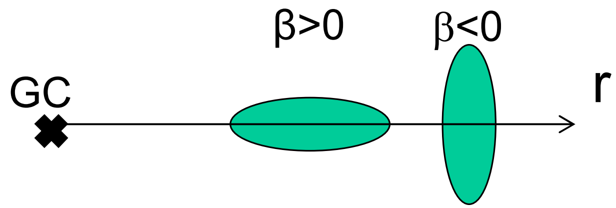
    $\beta>0$: radially anisotropic

    $\beta<0$: tangentially anisotropic

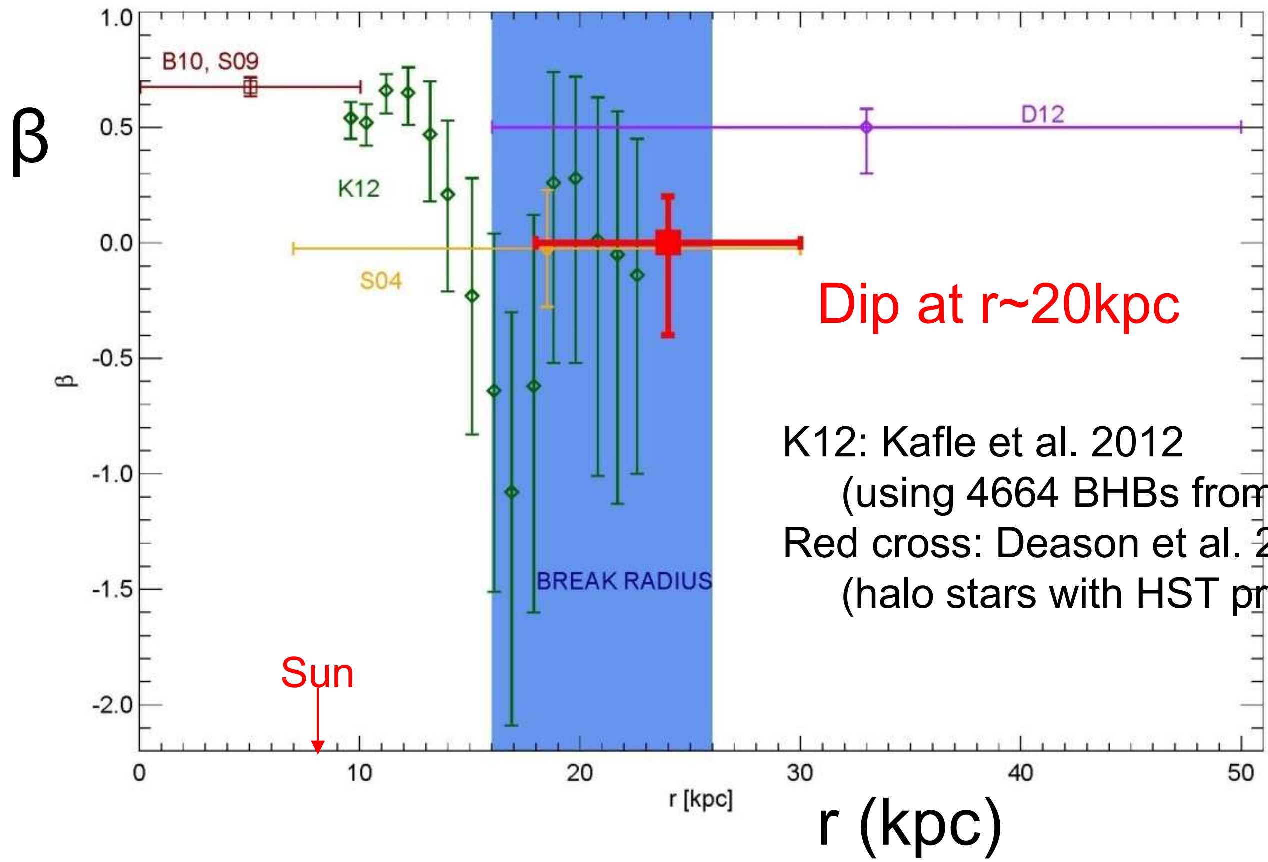


These velocity anisotropies reflect past merging/accretion histories

Velocity anisotropy parameter β(r)



Radial anisotropy:
$\beta(r)=1-(\sigma_\theta^2+\sigma_\phi^2)/2\sigma_r^2$



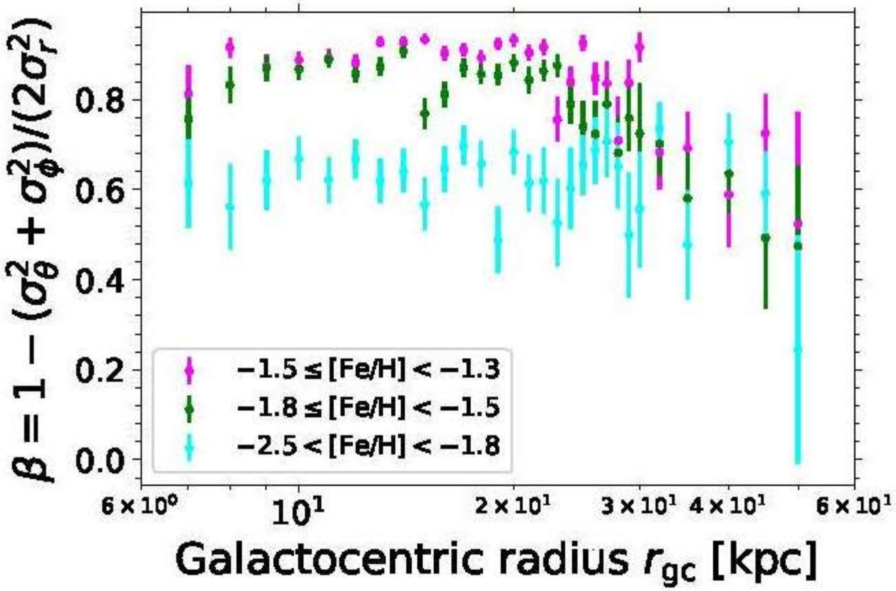
Dip at r~20kpc

K12: Kafle et al. 2012
(using 4664 BHBs from SDSS)
Red cross: Deason et al. 2013
(halo stars with HST proper motions)

Recent results on β(r)

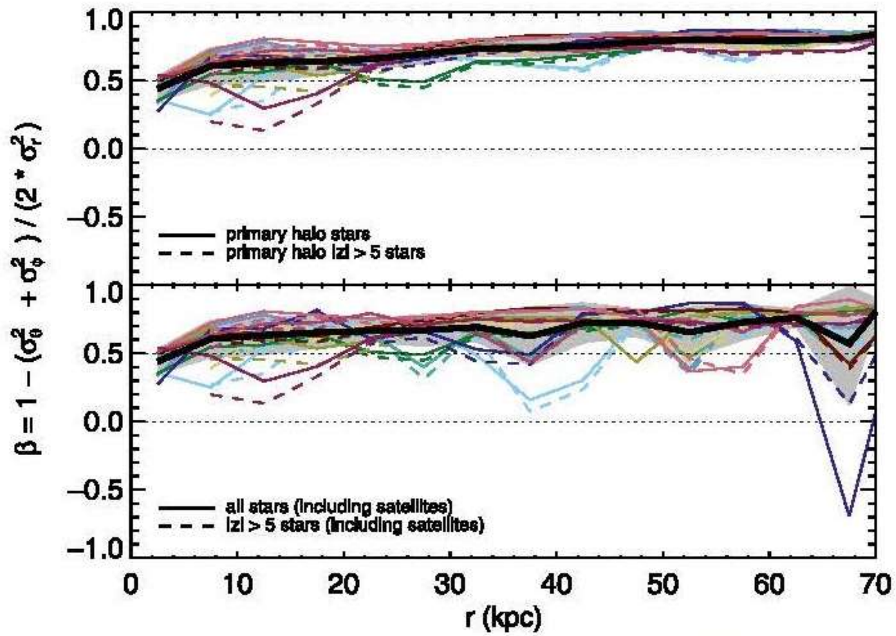
Bird et al. 2019
using 5600 K giants from
LAMOST and Gaia DR2

-1.8<[Fe/H]<-1.3: boundary
at r ~ 20kpc
[Fe/H]<-1.8: no dip

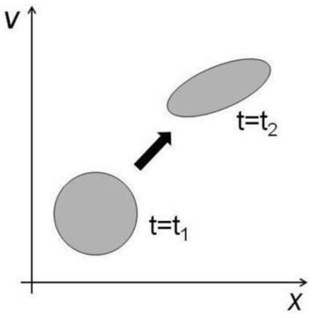


Loebman et al. 2019
using simulation results
by Bullock & Johnston 2005
(hierarchical clustering process)

β ~ 0.7
Radially anisotropic
over entire radii
Presence of temporal dips



4. Jeans equations



Continuity eq.
in phase space

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \frac{\partial (f \dot{x}_i)}{\partial x_i} + \sum_{i=1}^{3} \frac{\partial (f \dot{v}_i)}{\partial v_i} = 0$$

Boltzmann equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0$$

moment over v_i

$$\frac{\partial n}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(n \langle v_i \rangle) = 0$$

$$\frac{\partial n \langle v_j \rangle}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(n \langle v_i v_j \rangle) + n \frac{\partial \Phi}{\partial x_j} = 0$$

$$n(\boldsymbol{r}) = \int f d^3 \boldsymbol{v} \qquad \langle v_i \rangle = \frac{1}{n} \int v_i f d^3 \boldsymbol{v}$$

Jeans equations

$$n \frac{\partial \langle v_j \rangle}{\partial t} + \sum_{i=1}^{3} n \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -n \frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{3} \frac{\partial n \sigma_{ij}^2}{\partial x_i}$$

(hydrodynamical description)

$$\sigma_{ij}^2 = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$$

Jeans theorem

$$\frac{dI}{dt} = \boldsymbol{v} \cdot \nabla I - \nabla \Phi \cdot \frac{\partial I}{\partial \boldsymbol{v}} = 0$$

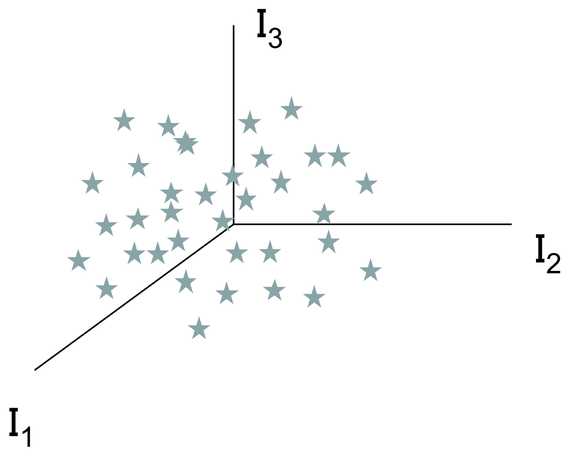
I is a solution to steady-state collisionless Boltzmann eq.

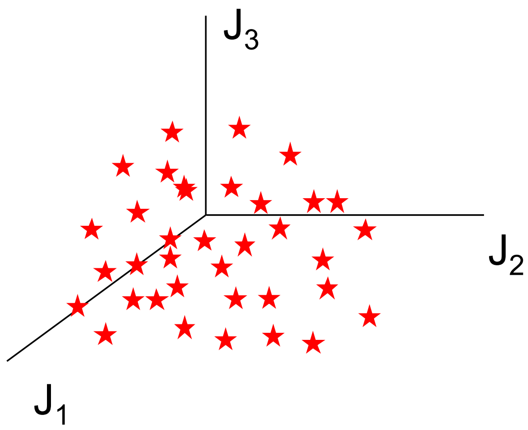
f(I(r,v)): a solution to steady-state collisionless Boltzmann eq.

・Strong Jeans Theorem

Potential Φ allowing only regular orbits
(no resonance among 3 orbital frequencies)
⇒ 3 isolating integrals
⇒ DF depends only these 3 integrals

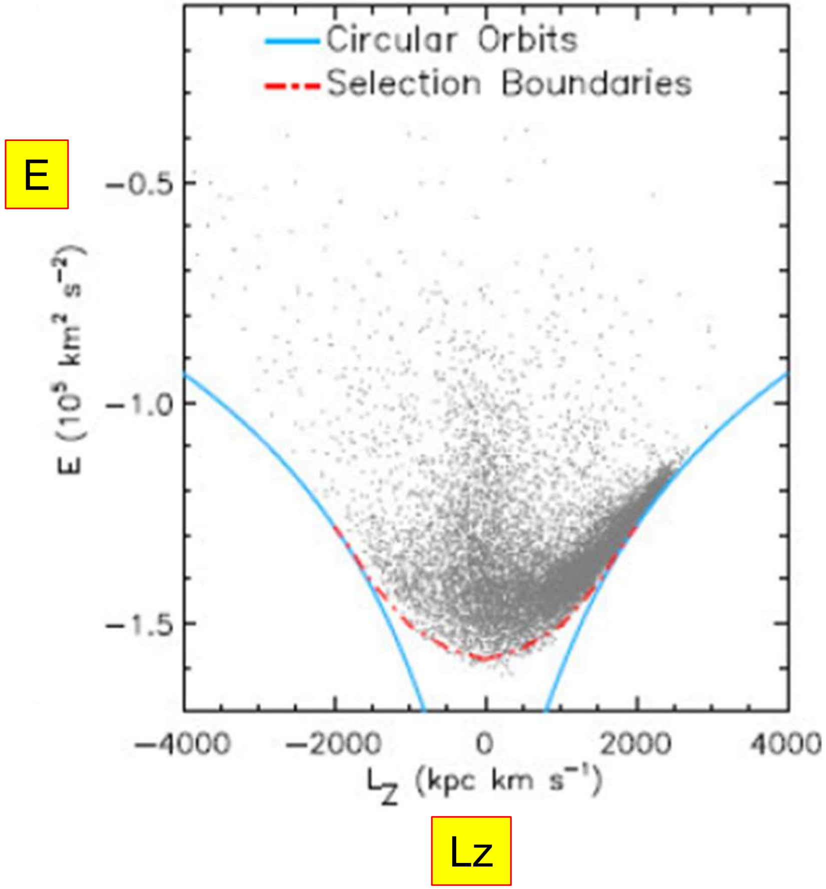
$f(I_1,I_2,I_3)$ $I_1=E$, $I_2=L_z^2/2$ $f(J_1,J_2,J_3)$ $J_i(I_i)$



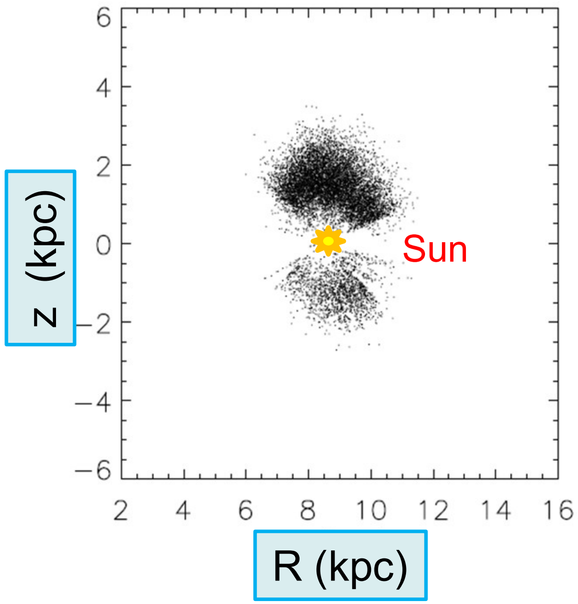


Nearby stars in (E, L_z, I_3) phase space
SDSS-DR7 Calibration Stars + Gaia DR2

Carollo & Chiba 2021

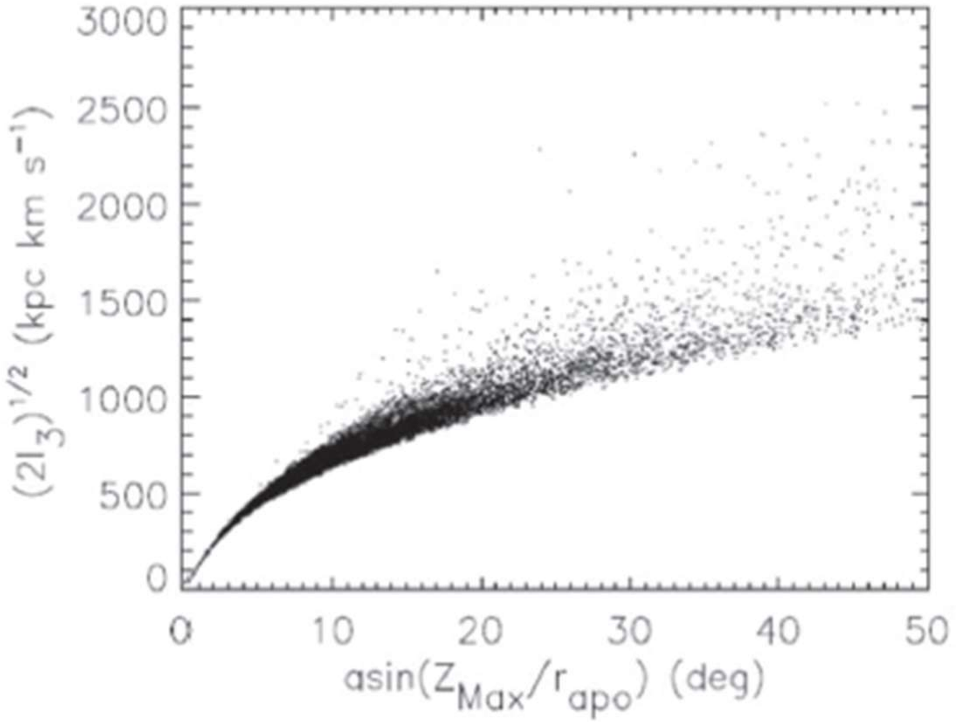


(R,z) distribution of sample stars

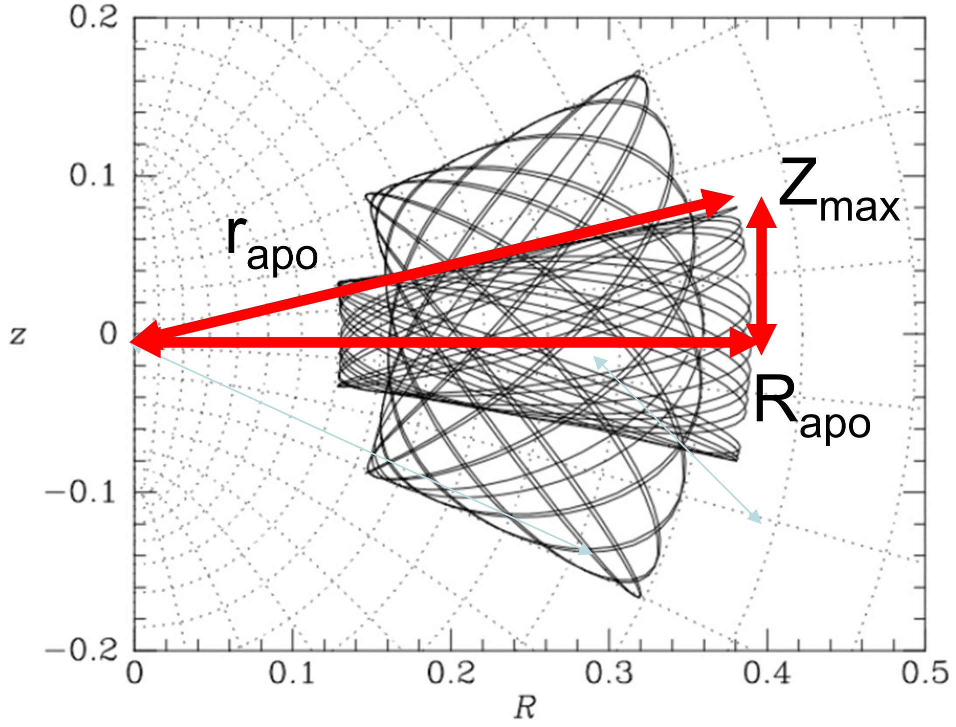


What is the third integral, I_3 ?

$(2I_3)^{1/2}$



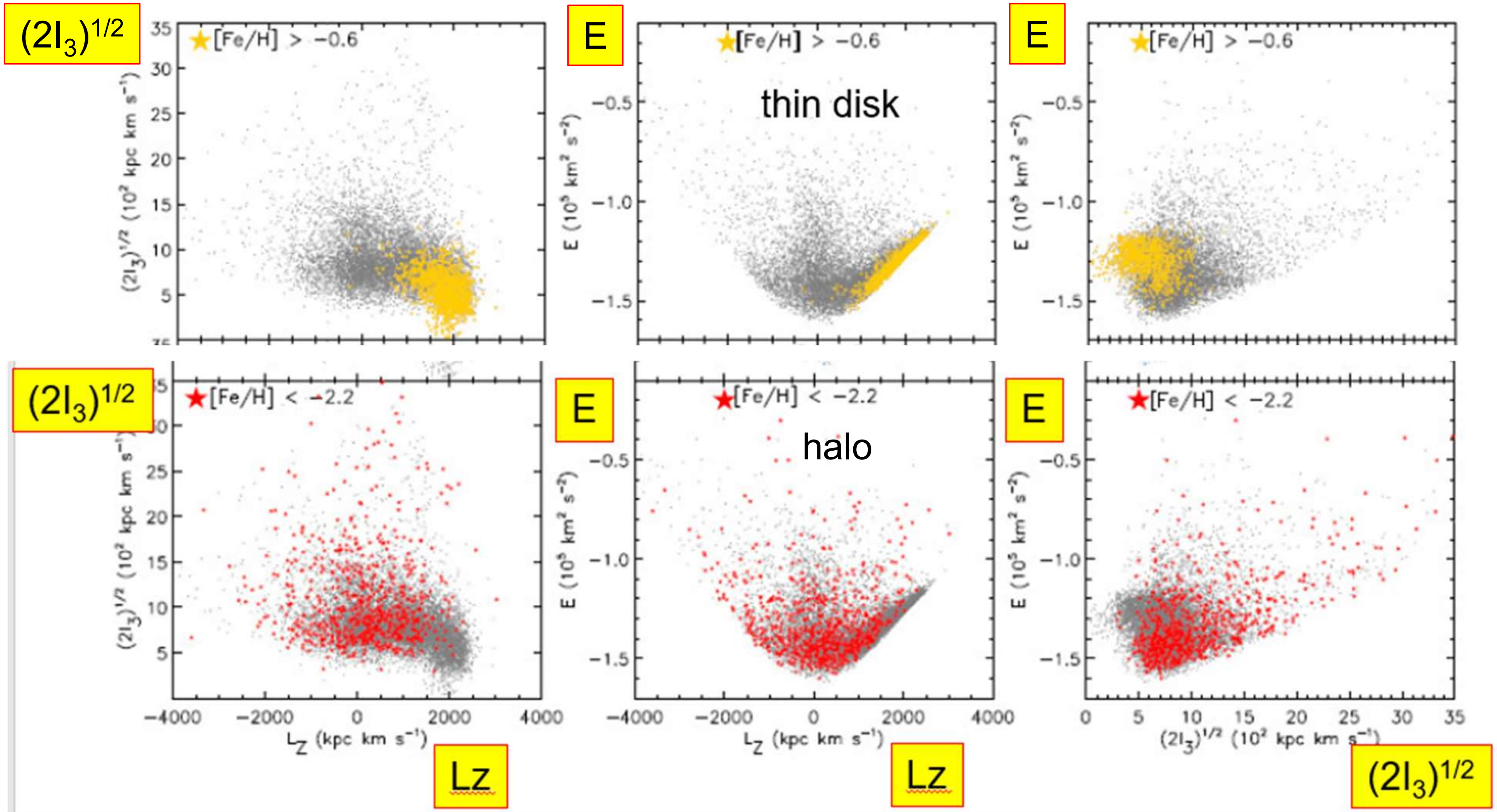
arcsin (Z_max/r_apo) (deg)
~ orbital inclination angle θ



Nearby stars in (E,L_z,I_3) phase space
SDSS-DR7 Calibration Stars + Gaia DR2

Carollo & Chiba 2021

(grey: all stars)



Simple cases for Jeans equations (I)

- Spherical system

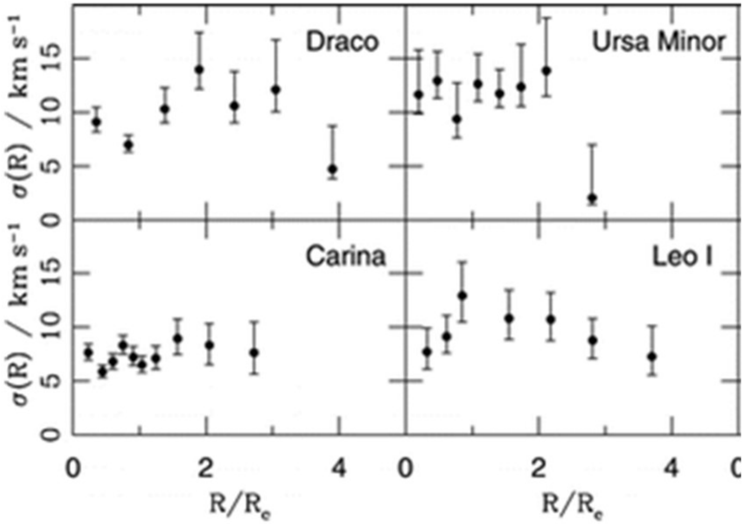
$$\frac{1}{n}\frac{dn\sigma_r^2}{dr} + 2\frac{\beta\sigma_r^2}{r} = -\frac{d\Phi}{dr} = -\frac{GM(r)}{r^2}$$

$$\beta \equiv 1 - (\sigma_\theta^2 + \sigma_\phi^2)/(2\sigma_r^2)$$

$$\beta = const. \Rightarrow \quad n\sigma_r^2 = r^{-2\beta}\int_r^\infty \frac{nGM(r')}{r'^2} r'^{2\beta} dr'$$

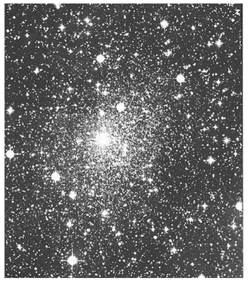
Example: Line-of-sight velocity dispersion profile in MW dwarf satellites

σ_{los}



Leo I

Car



$\sigma_{los}(R)$
$\sigma_r(r)$

M(r)
Dark matter

Simple cases for Jeans equations (II)

- Axisymmetric system

$$\frac{1}{n}\frac{\partial n\langle v_R^2\rangle}{\partial R} + \frac{1}{n}\frac{\partial n\langle v_R v_z\rangle}{\partial z} + \frac{\langle v_R^2\rangle - \langle v_\phi^2\rangle}{R} = -\frac{\partial \Phi}{\partial R}$$

$$\frac{1}{n}\frac{\partial n\langle v_R v_z\rangle}{\partial R} + \frac{1}{n}\frac{\partial n\langle v_z^2\rangle}{\partial z} + \frac{\langle v_R v_z\rangle}{R} = -\frac{\partial \Phi}{\partial z}\ .$$

Example 1: R direction

At z~0 $\partial n/\partial z = 0,\ \ \partial\langle v_R v_z\rangle/\partial z = 0$

Circular velocity $V_c^2 = R\partial\Phi/\partial R$

$$\langle v_\phi\rangle^2 = V_c^2 - \sigma_R^2\left[\frac{\sigma_\phi^2}{\sigma_R^2} - 1 - \frac{\partial \ln(n\sigma_R^2)}{\partial \ln R}\right]$$

Asymmetric drift

σ_R^2 is large (old stars) $\langle v_\phi\rangle$ < V_c

σ_R^2 is small (young stars) $\langle v_\phi\rangle \simeq V_c$

Example 2: z direction

At z~0 $\partial/\partial z$ >>1

$$\frac{1}{n}\frac{\partial n\sigma_z^2}{\partial z} = -\frac{\partial \Phi}{\partial z}$$

vertical equilibrium

5. Virial theorem

$$\frac{\partial n\langle v_j\rangle}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(n\langle v_i v_j\rangle) + n\frac{\partial \Phi}{\partial x_j} = 0$$

xmx_k $\quad \rho \equiv nm$

$$\int x_k \frac{\partial \rho\langle v_j\rangle}{\partial t} d^3x + \int x_k \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\langle v_i v_j\rangle) d^3x + \int \rho x_k \frac{\partial \Phi}{\partial x_j} d^3x = 0$$

$$\frac{1}{2}\frac{d^2 I_{jk}}{dt^2} = 2K_{jk} + W_{jk}$$

$$\begin{cases} I_{jk} \equiv \int \rho x_j x_k d^3x, \\ K_{jk} \equiv \frac{1}{2}\int \rho\langle v_j v_k\rangle d^3x, \\ W_{jk} = W_{kj} \equiv -\int \rho x_k \frac{\partial \Phi}{\partial x_j} d^3x \end{cases}$$

Steady state

$$2K + W = 0$$

$$\begin{cases} K = M\langle v^2\rangle/2 \\ W = -GM^2/R_g \end{cases} \Rightarrow \langle v^2\rangle = \frac{GM}{R_g} \simeq 0.4\frac{GM}{R_h} \Rightarrow M$$

R_g: gravitational radius

R_h: half-mass radius $\simeq 0.4R_g$