How is the local-scale gravitational instability influenced by the surrounding large-scale structure formation

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We develop the formalism to investigate the relation between the evolution of the large-scale (quasi) linear structure and that of the small-scale nonlinear structure in Newtonian cosmology within the Lagrangian framework. In doing so, we first derive the standard Friedmann expansion law using the averaging procedure over the present horizon scale. Then the large-scale (quasi) linear flow is defined by averaging the full trajectory field over a large-scale domain, but much smaller then the horizon scale. The rest of the full trajectory field is supposed to describe small-scale nonlinear dynamics. We obtain the evolution equations for the large-scale and small-scale part of the trajectory field. These are coupled each other in most general situations.

It is shown that if the shear deformation of fluid element is ignored in the averaged large-scale dynamics, the small-scale dynamics is described by Newtonian dynamics in an effective Friedmann-Robertson-Walker (FRW) background with a local scale factor. The local scale factor is defined by the sum of the global scale factor and the expansion deformation of the averaged large-scale displacement field. This means that the evolution of small-scale fluctuations is influenced by the surrounding large-scale structure through the modification of FRW scale factor. The effect might play an important role in the structure formation scenario. Furthermore, it is argued that the so-called *optimizing* or *truncated* Lagrangian perturbation theory is a good approximation in investigating the large-scale structure formation up to quasi nonlinear regime even when the small-scale fluctuations are in the non-linear regime.

Key words: Gravitational Instability, Newtonian Cosmology, Averaging Method, Large-scale structure of the Universe

1. INTRODUCTION

The observation of the anisotropy of the cosmic microwave background (CMB) indicates that the universe is remarkably isotropic on the present horizon scale. Thus it is natural to describe the horizon scale spatial geometry of the universe by a homogeneous and isotropic metric, namely, the Friedmann-Robertson-Walker (FRW) model. However, the real universe is neither isotropic nor homogeneous on local scales and has a hierarchical structure such as galaxies, clusters of galaxies, superclusters of galaxies and so on. It has been naively regarded that the FRW model is a large scale average of a locally inhomogeneous real universe. There have been several studies in this direction in general relativity [11–13,16,23,8].

Aside from such a fundamental problem, there is an interesting and practical problem associated with inhomogeneities across various scales. Can one ask if the formation of small-scale structures is influenced by the gravitational effect of structures with larger scales? Such a circumstance effect may be important and even essential to clarify the process of the hierarchical structure formation.

This is the problem we attack in the present paper. Namely, we develop the formalism to investigate the gravitational instability in general situations where the large-scale linear and the small-scale non-linear fluctuations coexist. If one uses the N-body simulation to answer the above question, one needs high spatial resolution over very large box comparable with the horizon scale which may be well above the ability of the present computer. However, it seems reasonable to regard the situation such as local nonlinear structures are overlapped on a smoothed large-scale linear structure and the large-scale dynamics may well be treated by Zel'dovich-type approximation. This suggests us to adopt an analytical approach based on the Lagrangian perturbation theory in Newtonian cosmology.

The reason why we consider Newtonian cosmology is partly because of its simplicity and partly because the Newtonian cosmology is a good approximation to a realistic inhomogeneous universe. In fact we have shown that the

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Newtonian cosmology in the relativistic framework is a good approximation even for the perturbations not only inside but also beyond the present horizon scale [25].

The reason why we work within the Lagrangian framework is because it seems easy to introduce the averaging process. This is essential in our formalism because we will going to define the global expansion law as well as the large-scale smooth trajectory field defined by averaging. In fact Buchert and Ehlers studied the averaging problem in Newtonian cosmology in the Lagrangian framework by performing spatial averages of Eulerian kinematical fields such as the rate of expansion $\theta := \nabla_x \cdot \boldsymbol{v}$ of the fluid flow [7,10]. They have found that the background introduced by the spatial averaging obeys the FRW cosmology under the appropriate assumption that the peculiar velocity field, which is defined as a deviation from a Hubble flow in the Eulerian picture, obeys the periodic boundary condition on a sufficiently large scale (see below).

In this paper we take a different approach from Buchert & Ehlers, and work entirely within the Lagrangian framework. Namely, we divide the trajectory field into mean flow and deviation field, and then take the spatial average of Lagrange-Newton system in order to introduce the horizon scale background as well as the large-scale averaged trajectory field. We arrive at the same conclusion as that of Buchert & Ehlers when averaged over the horizon scale. Then, in order to separate the non-linear dynamics from the large-scale dynamics, we further separate the deviation field into two parts, the averaged large-scale field and the rest. The evolution equation for the large-scale field is then obtained by averaging the local dynamical equation over a large domain smaller than the present horizon scale in which the periodic boundary condition is applied for the small-scale perturbations. In this way we will obtain the evolution equations for the averaged large-scale field and the local-scale field. Then the evolution equation for the small-scale trajectory field which describes the local inhomogeneities is obtained by subtracting the evolution equation of the averaged large-scale trajectory field from that of the full trajectory field. The evolution equation for the local-scale field is naturally come out by subtracting the averaged evolution equation for the large-scale field from the non-averaged equation. These equations couple each other in general situation. In this way, we are able to study how the smoothed large-scale structure is formed when the universe has non-linear structures on small scales as well as how the small-scale fluctuations grow in the surrounding environment.

This paper is organized as follows. In Section 2, we shall write down the basic equations needed in our considerations in the Lagrangian formalism developed by Buchert. In Section 3 we investigate the average properties of the equations derived in Section 2. We will have the averaged FRW background under the periodic boundary condition over the horizon scale. In Section 4, we develop the formalism to have evolution equation for the large-scale and small-scale fluctuations where the results in Section 3 are used frequently. In general the evolution of the large-scale fluctuations is influenced by the existence of the small-scale nonlinearity. We clarify the situations where the large-scale fluctuation behaves independently of small-scale structures. In those cases the evolution of the large-scale fluctuations can be described by the so-called "truncated" or "optimizing" Lagrangian perturbation theory which has been originally developed by many authors [9,18,19,26] in order to avoid the shell-crossing problem of nonlinearity on small scales. It is then shown that the small-scale dynamics is governed by the modified scale factor. The final section contains discussions. Throughout this paper, Latin indices take $1, 2, 3$, respectively.

2. BASIC EQUATIONS IN THE LAGRANGIAN PICTURE

Let us start with the basic system of equations in Newtonian cosmology describing the motion of a self-gravitating pressureless fluid so-called "dust". The dynamics of the fluid obeys the following familiar *Euler-Newton* system of equations in Newtonian hydrodynamics:

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho \boldsymbol{v}) = 0, \tag{2.1a}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla_x)\boldsymbol{v} = \boldsymbol{g}, \tag{2.1b}$$

$$\nabla_x \times \boldsymbol{g} = \boldsymbol{0}, \tag{2.1c}$$

$$\nabla_x \cdot \boldsymbol{g} = -4\pi G\rho + \Lambda c^2, \tag{2.1d}$$

where $\rho(\boldsymbol{x}, t)$, $\boldsymbol{v}(\boldsymbol{x}, t)$, and $\boldsymbol{g}(\boldsymbol{x}, t)$ denote the fields of mass density, velocity, and gravitational acceleration, respectively. The Poisson equation (2.1d) is extended including the cosmological constant for the sake of generality.

Following the Lagrangian formulation developed by Buchert [6], we concentrate on the integral curves $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{X}, t)$ of the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$:

$$\frac{d\boldsymbol{f}}{dt}\left(= \dot{\boldsymbol{f}}\right) := \left.\frac{\partial \boldsymbol{f}}{\partial t}\right|_X = \boldsymbol{v}(\boldsymbol{f}, t), \qquad \boldsymbol{f}(\boldsymbol{X}, t_I) \equiv \boldsymbol{X}, \tag{2.2}$$

where $\boldsymbol{X}$ denote the Lagrangian coordinates which label fluid elements, $\boldsymbol{x}$ are the positions of these elements in Eulerian space at time t, and t_I is the initial time when Lagrangian coordinates are defined.

Then we can express the field ρ, $\boldsymbol{v}$ and $\boldsymbol{g}$ in the Eulerian picture in terms of the Lagrangian coordinates $(\boldsymbol{X},t)$ [6,3] from Eqs.(2.1a), (2.2), and (2.1b), respectively:

$$\rho(\boldsymbol{X},t) = \frac{\mathring{\rho}\,(\boldsymbol{X})}{J(\boldsymbol{X},t)}, \tag{2.3a}$$

$$\boldsymbol{v}(\boldsymbol{X},t) = \dot{f}(\boldsymbol{X},t), \tag{2.3b}$$

$$\boldsymbol{g}(\boldsymbol{X},t) = \ddot{f}(\boldsymbol{X},t), \tag{2.3c}$$

where where J is the determinant of the deformation field $f_{i|j}$ (the vertical slash in the subscript denotes partial derivative with respect to the Lagrangian coordinate $\boldsymbol{X}$)and the quantities with $\circ$ such as $\mathring{\rho}$ denote the quantities at the initial time t_I henceforth and we used the fact $\mathring{J}= 1$. Thus, the continuity equation (2.1a) can be exactly integrated along the flow lines of the fluid elements in the Lagrangian picture [2,3,6]. As a result, dynamical variable in the Lagrangian picture is only a trajectory field $\boldsymbol{f}$. The equations (2.3b) and (2.3c) are like a point mechanics. The constraint equations (2.1c) and (2.1d) of the acceleration field $\boldsymbol{g}$ give us the four evolution equations of the single dynamical field $\boldsymbol{f}$ after the usual procedure in the Lagrangian formalism:

$$\epsilon_{abc} f_{i|a} f_{j|b} \ddot{f}_{j|c} = 0, \tag{2.4}$$

$$\frac{1}{2}\epsilon_{j\,ab}\epsilon_{icd} f_{c|a} f_{d|b} \ddot{f}_{i|j} - \Lambda c^2 J = -4\pi G\, \mathring{\rho}\,(\boldsymbol{X}), \tag{2.5}$$

where we used the mass conservation (2.3a). The set of equations can be solved in principle for some initial conditions if we give the initial density field $\mathring{\rho}(\boldsymbol{X})$ as a source function.

The system of equations (2.3a-c), (2.4), and (2.5) is the so-called "*Lagrange-Newton*" system and equivalent to the Euler-Newton system as long as the mapping $\boldsymbol{f}_t : \boldsymbol{X} \mapsto \boldsymbol{x}$ is invertible. Buchert has solved the above set of equations perturbatively taking the solutions of FRW models as the zero-th order background solution (discussed below) [2–4].

For the purpose of the later discussion, we consider the the Eulerian vorticity field $\omega_i = (1/2)\epsilon_{ijk} v_{k,j}$ and derive the *Helmholtz-Kelvin vorticity transport equation* in the Lagrangian representation [2,6]. First, from (2.3b) we can rewrite the vorticity field $\boldsymbol{\omega}$ in terms of the trajectory field $\boldsymbol{f}$ as

$$\omega_i = \frac{1}{2J}\epsilon_{abc} f_{j|a} f_{i|b} \dot{f}_{j|c}. \tag{2.6}$$

To derive the theorem, we need Eq.(2.4). Multiplying the equation by the $h_{l,i}$ whose inverse is $f_{i|a}$, we can obtain

$$\epsilon_{lbc} f_{j|b} \ddot{f}_{j|c} = 0. \tag{2.7}$$

This equation is rewritten as

$$\frac{d}{dt}\left(\epsilon_{lbc} f_{j|b} \dot{f}_{j|c}\right) = 0, \tag{2.8}$$

so it can be integrated exactly along the trajectory field:

$$\frac{1}{2}\epsilon_{blc} f_{j|b} \dot{f}_{j|c} = \mathring{\omega}_l, \tag{2.9}$$

where we used the initial condition $\mathring{\omega}_i= (1/2)\epsilon_{ijk}\dot{f}_{k|j}(\boldsymbol{X},t_I)$ from Eq.(2.6). Finally, multiplying the above equation (2.9) by the deformation field $f_{i|l}/J$ and using Eq.(2.6), we can obtain the following Helmholtz-Kelvin vorticity transport equation we are looking for:

$$\omega_i(\boldsymbol{X},t) = \frac{1}{J} f_{i|j}\ \mathring{\omega}_j\,(\boldsymbol{X}). \tag{2.10}$$

The vorticity field in the Eulerian picture evolves according to the above equation along the flow lines and is coupled to the density enhancement, because the field is proportional to the inverse of the determinant of the deformation field as the mass conservation equation (2.3a) [2]. This equation also means that if the initial vorticity field is zero at some point, the vorticity field remains zero at any time along the flow lines. Conversely, as the density field develops singularities ($J \to 0$), the vorticity field will blow up simultaneously even if the initial vorticity field is much smaller than the irrotational part and is not zero [2].

3. AVERAGING NEWTONIAN COSMOLOGIES

A. A Hubble flow for a trajectory field

Before discussing the averaging problem, we first consider the properties of trajectory field $\boldsymbol{f}$ in the Lagrange-Newton system derived in the previous section. We have reduced the description of the dynamics of any Eulerian field to the problem of finding the field of trajectories $\boldsymbol{f}$ as a solution of the Lagrange-Newton system (2.4) and (2.5). As in the Eulerian case, we are not able to write down any exact solution for generic initial data $\mathring{\rho}\ (\boldsymbol{X})$ without assuming a symmetry like plane or spherical symmetry. We may start with the simplest class of solutions, the homogeneous-isotropic ones, and then move on to the treatment of inhomogeneities.

Those fluid motions which are locally isotropic in the sense that, at any time and for each particle $\mathcal{P}$, there exists an neighborhood on which the field of velocities relative to $\mathcal{P}$ is invariant under all rotations about $\mathcal{P}$, are given by the following form with our choice (2.2) of Lagrangian coordinate:

$$\boldsymbol{x} = \boldsymbol{f}_H(\boldsymbol{X},t) = \frac{a(t)}{\mathring{a}}\boldsymbol{X}, \qquad \mathring{a} := a(t_I), \tag{3.1}$$

if we conventionally put $\boldsymbol{f}_H(\boldsymbol{0},t) = \boldsymbol{0}$. Such a flow is a well-known Hubble flow. Inserting this ansatz into the Lagrange-Newton system (2.4), (2.5) and the mass conservation (2.3a) yield the usual Friedmann equations

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\frac{\mathring{\rho}_b\mathring{a}^3}{a^3(t)} + \frac{\Lambda c^2}{3}, \tag{3.2}$$

$$\rho_b(t) = \frac{\mathring{a}^3}{a^3(t)}\,\mathring{\rho}_b\ . \tag{3.3}$$

Thus, the quantity $a(t)$ agrees with the scale factor in FRW cosmology. It should be remarked that the assumption of homogeneous and isotropic matter flow (3.1) makes the initial density independent of $\boldsymbol{X}$ via the equation (2.5): $\mathring{\rho}\,(\boldsymbol{X}) \equiv \mathring{\rho}_b =$ constant. In other words, the existence of the fluctuation for the initial density field produces no longer a Hubble flow such as (3.1). We may use it to integrate Eq.(3.2) yielding Friedmann's differential equation:

$$\frac{\dot{a}^2}{a^2} + \frac{\mathcal{K}c^2}{a^2} = \frac{8\pi G}{3}\frac{\mathring{\rho}_b\mathring{a}^3}{a^3} + \frac{\Lambda c^2}{3}; \qquad \mathcal{K} = \text{const.} \tag{3.4}$$

where $\mathcal{K}$ is the constant of integration mathematically and can be regarded as the curvature parameter of the FRW model. Naturally, Eq.(2.6) gives

$$\boldsymbol{\omega} = \boldsymbol{0}. \tag{3.5}$$

Thus the assumption (3.1) for the trajectory field produces the standard Friedmann cosmologies.

B. Averaged properties of the Lagrange-Newton system in an inhomogeneous universe

We now consider the trajectory field when there exist inhomogeneities in the universe. In the application of Lagrangian theory to the averaging problem in cosmology, we examine the behavior of some spatially compact domain $\mathcal{D}(t)$ on the Eulerian space occupied by the fluid elements, which corresponds to the initial domain $\mathring{\mathcal{D}}$ of the Lagrangian coordinates via the mapping $\boldsymbol{f}_t : \boldsymbol{X} \mapsto \boldsymbol{x}$.

For our purpose, we set the average flow in the form of a Hubble flow and define the (not necessarily small) deviation field $\boldsymbol{P}$ from the average flow without loss of generality, so that the full trajectory field $\boldsymbol{f}$ of an inhomogeneous model reads:

$$\boldsymbol{f}(\boldsymbol{X},t) := \boldsymbol{f}_H^{\mathcal{D}}(\boldsymbol{X},t) + \boldsymbol{P}(\boldsymbol{X},t) = \frac{a_{\mathcal{D}}(t)}{\mathring{a}_{\mathcal{D}}}\boldsymbol{X} + \boldsymbol{P}(\boldsymbol{X},t); \qquad \boldsymbol{P}(\boldsymbol{X},t_I) := \boldsymbol{0}, \tag{3.6}$$

For the sake of convenience, we here consider the trajectory field in the form of Zel'dovich type solution [27]. However, it should be noted that we can define the full trajectory field by Eq.(3.6) for any fluid elements in an arbitrary initial domain $\mathring{\mathcal{D}}$, and the scale factor depends naturally on the chosen domain. We also remark that since we have imposed

no condition on the deviation field $\boldsymbol{P}$, this consideration is applied to the non-linear situations when the density contrast field are larger than unity.

Using the full trajectory field (3.6), we obtain by a little computation the following expression for the determinant J of deformation field $f_{i|j}$:

$$\begin{aligned} J &= \frac{1}{3!}\epsilon_{abc}\epsilon_{ijk}f_{a|i}f_{b|j}f_{c|k} \\ &= \frac{a_{\mathcal{D}}^3}{\mathring{a}_{\mathcal{D}}^3} + \frac{a_{\mathcal{D}}^2}{\mathring{a}_{\mathcal{D}}^2}P_{i|i} + \frac{1}{2}\frac{a_{\mathcal{D}}}{\mathring{a}_{\mathcal{D}}}\left(P_iP_{j|j} - P_jP_{i|j}\right)_{|i} + \frac{1}{6}\left[P_i\left(P_{j|j}P_{k|k} - P_{j|k}P_{k|j}\right) + 2P_j\left(P_{i|k}P_{k|j} - P_{k|k}P_{i|j}\right)\right]_{|i} \\ &:= \frac{1}{\mathring{a}_{\mathcal{D}}^3}\left(a_{\mathcal{D}}^3(t) + \tilde{\mathcal{J}}_{i|i}(\boldsymbol{X},t)\right), \end{aligned} \tag{3.7}$$

where

$$\tilde{\mathcal{J}}_i := \mathring{a}_{\mathcal{D}}\, a_{\mathcal{D}}^2 P_i + \frac{\mathring{a}_{\mathcal{D}}^2}{2}a_{\mathcal{D}}\left(P_iP_{j|j} - P_jP_{i|j}\right) + \frac{\mathring{a}_{\mathcal{D}}^3}{6}\left[P_i\left(P_{j|j}P_{k|k} - P_{j|k}P_{k|j}\right) + 2P_j\left(P_{i|k}P_{k|j} - P_{k|k}P_{i|j}\right)\right]. \tag{3.8}$$

Note that the second term on the right-hand-side of Eq.(3.7) is expressed by the divergence of the vector $\tilde{\boldsymbol{\mathcal{J}}}(\boldsymbol{X},t)$ with respect to the Lagrangian coordinate and it is defined in terms of the deviation displacement vector $\boldsymbol{P}$. Since the volume elements at t and t_I are related by $d^3x = \mathcal{J}d^3X$, we use the above equation to rewrite the volume $V_{\mathcal{D}}(t)$ of the domain $\mathcal{D}(t)$ of the fluid in the following form:

$$\begin{aligned} V_{\mathcal{D}}(t) := \int_{\mathcal{D}(t)} d^3x &= \int_{\mathring{\mathcal{D}}} d^3X\mathcal{J}(\boldsymbol{X},t) \\ &= \frac{1}{\mathring{a}_{\mathcal{D}}^3}\left(a_{\mathcal{D}}^3 + \left\langle\tilde{\mathcal{J}}_{i|i}\right\rangle_{\mathring{\mathcal{D}}}\right)\mathring{V}_{\mathcal{D}} \\ &= \frac{a_{\mathcal{D}}^3}{\mathring{a}_{\mathcal{D}}^3}\,\mathring{V}_{\mathcal{D}} + \frac{1}{\mathring{a}_{\mathcal{D}}^3}\int_{\partial\mathring{\mathcal{D}}} d\boldsymbol{S}_X\cdot\tilde{\boldsymbol{\mathcal{J}}}(\boldsymbol{X},t), \end{aligned} \tag{3.9}$$

where we have applied Gauß's theorem to transform the volume integral to a surface integral over the boundary $\partial\mathring{\mathcal{D}}$ of the initial domain:

$$\int_{\mathring{\mathcal{D}}} d^3X\nabla_X\cdot\tilde{\boldsymbol{\mathcal{J}}} = \int_{\partial\mathring{\mathcal{D}}} d\boldsymbol{S}_X\cdot\tilde{\boldsymbol{\mathcal{J}}}.$$

In the derivation of Eq.(3.9), $\langle\ldots\rangle_{\mathring{\mathcal{D}}}$ denotes the spatial average of a tensor field over the initial domain $\mathring{\mathcal{D}}$, and we regard $a_{\mathcal{D}}(t)$ as the scale factor of that domain $\mathcal{D}(t)$ in the Eulerian space. The quantity $\mathring{V}_{\mathcal{D}}$ denotes the volume of initial domain $\mathring{\mathcal{D}}$ considered,

$$\mathring{V}_{\mathcal{D}} := \int_{\mathring{\mathcal{D}}} d^3X. \tag{3.10}$$

Buchert & Ehlers [7] have used the domain dependent scale factor $a_{\mathcal{D}}$ defined by $V_{\mathcal{D}} \equiv a_{\mathcal{D}}^3(t)$, but Eq.(3.9) means that such a scale factor does not agree with our scale factor defined by Eq.(3.6). Likewise, the spatial average of the density field (2.3a) over the domain $\mathcal{D}$ in Eulerian space at any time t may be calculated as follows.

$$\begin{aligned} \langle\rho\rangle_{\mathcal{D}}(t) &= \frac{1}{V_{\mathcal{D}}}\int_{\mathcal{D}(t)} d^3x\frac{\mathring{\rho}\,(\boldsymbol{X})}{\mathcal{J}(\boldsymbol{X},t)} \\ &= \frac{1}{V_{\mathcal{D}}}\int_{\mathring{\mathcal{D}}} d^3X\,\mathring{\rho} \\ &= \frac{\mathring{V}_{\mathcal{D}}}{V_{\mathcal{D}}(t)}\left\langle\mathring{\rho}\right\rangle_{\mathring{\mathcal{D}}} \\ &= \frac{\mathring{a}_{\mathcal{D}}^3}{a_{\mathcal{D}}^3 + \left(1/\,\mathring{V}_{\mathcal{D}}\right)\int_{\partial\mathring{\mathcal{D}}} d\boldsymbol{S}_X\cdot\tilde{\boldsymbol{\mathcal{J}}}}\left\langle\mathring{\rho}\,(\boldsymbol{X})\right\rangle_{\mathring{\mathcal{D}}}. \end{aligned} \tag{3.11}$$

Next we consider the average of the dynamical equation (2.5) of the trajectory field. Likewise, the first term on the left-hand-side of Eq.(2.5) can be rewritten by inserting Eq.(3.6) in the following form:

$$\frac{1}{2}\epsilon_{abc}\epsilon_{ijk}f_{a|i}f_{j|b}\ddot{f}_{k|c} = \frac{1}{\mathring{a}_{\mathcal{D}}^3}\left(3a_{\mathcal{D}}^2\ddot{a}_{\mathcal{D}} + \mathcal{Q}_{i|i}(\boldsymbol{X},t)\right), \tag{3.12}$$

where

$$\begin{aligned}\mathcal{Q}_i(\boldsymbol{X},t) :=& \mathring{a}_{\mathcal{D}}\, a_{\mathcal{D}}^2\left(\ddot{P}_i + 2\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}P_i\right)\\ &+ \mathring{a}_{\mathcal{D}}^2\, a_{\mathcal{D}}\left(\ddot{P}_iP_{j|j} - \ddot{P}_jP_{i|j}\right) + \frac{\mathring{a}_{\mathcal{D}}^2}{2}\ddot{a}_{\mathcal{D}}\left(P_iP_{j|j} - P_jP_{i|j}\right)\\ &+\frac{\mathring{a}_{\mathcal{D}}^3}{2}\left[\ddot{P}_i\left(P_{j|j}P_{k|k} - P_{j|k}P_{k|j}\right) + 2\ddot{P}_j\left(P_{i|k}P_{k|j} - P_{i|j}P_{k|k}\right)\right],\end{aligned} \tag{3.13}$$

and we again remark that the second term on the right-hand-side of Eq.(3.12) can also be expressed by the divergence of the vector $\boldsymbol{\mathcal{Q}}(\boldsymbol{X},t)$. Thus, we can get the local evolution equation for the domain dependent scale factor $a_{\mathcal{D}}$ and the displacement vector P_i:

$$3a_{\mathcal{D}}^2\ddot{a}_{\mathcal{D}} + 4\pi G\,\mathring{\rho}(\boldsymbol{X})\,\mathring{a}_{\mathcal{D}}^3 - \Lambda c^2 a_{\mathcal{D}}^3 = -\mathcal{Q}_{i|i} + \Lambda c^2\mathcal{J}_{i|i}. \tag{3.14}$$

Averaging over the initial domain $\mathring{\mathcal{D}}$ of the above equation (3.14) leads to the following equation:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\frac{\left\langle\mathring{\rho}\,(\boldsymbol{X})\right\rangle_{\mathring{\mathcal{D}}}\mathring{a}_{\mathcal{D}}^3}{a_{\mathcal{D}}^3} - \Lambda c^2 = \frac{1}{a_{\mathcal{D}}^3\,\mathring{V}_{\mathcal{D}}}\int_{\partial\mathring{\mathcal{D}}} d\boldsymbol{S}_X\cdot\left(-\boldsymbol{\mathcal{Q}} + \Lambda c^2\tilde{\boldsymbol{\mathcal{J}}}\right), \tag{3.15}$$

where we again used Gauß's theorem. If the local equation (3.14) and the averaged equation (3.15) can be solved simultaneously, the domain dependent scale factor $a_{\mathcal{D}}(t)$ and the local displacement vector P_i are obtained in principle, respectively. This averaged equation can also be interpreted as a standard Friedmann equation for the "effective mass density" $\rho_{\rm eff}$ defined by,

$$4\pi G\rho_{\rm eff}(t) := 4\pi G\frac{\left\langle\mathring{\rho}\right\rangle_{\mathring{\mathcal{D}}}\mathring{a}_{\mathcal{D}}^3}{a_{\mathcal{D}}^3} + \frac{1}{a_{\mathcal{D}}^3\,\mathring{V}_{\mathcal{D}}}\int_{\partial\mathring{\mathcal{D}}} d\boldsymbol{S}_X\cdot\left(\boldsymbol{\mathcal{Q}} - \Lambda c^2\tilde{\boldsymbol{\mathcal{J}}}\right), \tag{3.16}$$

where the first term on the right-hand-side decreases clearly in proportional to $a_{\mathcal{D}}^{-3}$. The equation (3.15) shows that inhomogeneities have an accelerating effect on the expansion rate $\dot{a}_{\mathcal{D}}/a_{\mathcal{D}}$ of the average flow, if the term $\int_{\partial\mathring{\mathcal{D}}} d\boldsymbol{S}_X\cdot(-\boldsymbol{\mathcal{Q}})$ on the right-hand-side dominates the other terms and is positive. This shows that the evolution of the domain dependent scale factor $a_{\mathcal{D}}$ does in fact depend on the chosen domain, that is, the averaged expansion will be different from the usual Friedmann laws (3.2) if the average involving $\boldsymbol{\mathcal{Q}}$ and $\tilde{\boldsymbol{\mathcal{J}}}$ do not vanish.

Next, we consider the property of the average of the vorticity field $\boldsymbol{\omega}$. Similarly, performing the average of Eq.(2.10) over the domain $\mathcal{D}$ on the Eulerian space, we can obtain

$$\langle\omega_i\rangle_{\mathcal{D}} = \frac{\mathring{a}_{\mathcal{D}}^3}{a_{\mathcal{D}}^3(t) + \left(1/\,\mathring{V}_{\mathcal{D}}\right)\int_{\partial\mathring{\mathcal{D}}} d\boldsymbol{S}_X\cdot\tilde{\boldsymbol{\mathcal{J}}}(\boldsymbol{X},t)}\left[\frac{a_{\mathcal{D}}}{\mathring{a}_{\mathring{\mathcal{D}}}}\left\langle\mathring{\omega}_i\right\rangle_{\mathring{\mathcal{D}}} + \frac{1}{\mathring{V}_{\mathcal{D}}}\int_{\partial\mathring{\mathcal{D}}} dS_{Xj}\left(P_i\,\mathring{\omega}_j\right)\right]. \tag{3.17}$$

Noting that the quantity $\mathring{\omega}_i$ is divergence-less by definition, namely $\mathring{\omega}_{i|i} = (1/2)\epsilon_{ijk}\,\mathring{v}_{k|ji} = 0$, we have again used Gauß's theorem in rewriting the second term on the right-hand-side. The equation (3.17) means that if $\mathring{\omega}_i$ vanishes at every point on the initial hypersurface, it leads to $\langle\omega_i\rangle_{\mathcal{D}} = 0$ at any time.

Thus, the average properties of the Lagrange-Newton system do not necessarily agree with the FRW cosmologies in general inhomogeneous universe. However, based on the observation of extreme isotropy of the cosmic microwave background radiation, we expect that the universe is almost isotropic and homogeneous on a sufficiently large scale. We consider how this fact is expressed mathematically in terms of averaged variables.

As discussed above, we could define the displacement vector $\boldsymbol{P}$ as representing the deviation from the mean flow generated by the inhomogeneities. The resultant equation (3.15) then shows us how this field $\boldsymbol{P}$ determines the

backreaction on the scale $a_{\mathcal{D}}(t)$ from Friedmann's law. Note that the backreaction terms are expressed by the surface integrals over the boundary of the initial domain $\partial\mathring{\mathcal{D}}$. Thus if we employ the periodic boundary condition for the deviation vector field $\boldsymbol{P}$ on some sufficiently large scale $\mathcal{D}_p$, the backreaction terms in Eq.(3.15) are exactly zero:

$$3\frac{\ddot{a}_{\mathcal{D}_p}}{a_{\mathcal{D}_p}}+4\pi G\frac{\mathring{\rho}_b\mathring{a}^3_{\mathcal{D}_p}}{a^3_{\mathcal{D}_p}}-\Lambda c^2=0, \tag{3.18}$$

with

$$\mathring{\rho}_b\equiv\left\langle\mathring{\rho}\left(\boldsymbol{X}\right)\right\rangle_{\mathring{\mathcal{D}}_p}=\text{constant}. \tag{3.19}$$

Similarly, Eq.(3.11) gives

$$\rho_b(t)\equiv\langle\rho(\boldsymbol{X},t)\rangle_{\mathcal{D}_p(t)}=\frac{\mathring{\rho}_b\mathring{a}^3_{\mathcal{D}_p}}{a^3_{\mathcal{D}_p}(t)}. \tag{3.20}$$

We thus obtain the usual definitions of the homogeneous background density field (3.19) and (3.20) using the spatial averaging. Since we have not restricted ourselves to the perturbative situation where the deviation vector $\boldsymbol{P}$ is infinitesimally small, this discussion is always valid for non-linear situations under the periodic boundary condition.

Under the same periodic assumption, Eq.(3.17) gives us the following form as the average of the Eulerian vorticity field:

$$\langle\omega_i\rangle_{\mathcal{D}_p}=\frac{\mathring{a}^2_{\mathcal{D}_p}}{a^2_{\mathcal{D}_p}}\left\langle\mathring{\omega}_i\right\rangle_{\mathring{\mathcal{D}}_p} \tag{3.21}$$

with the scale factor $a_{\mathcal{D}_p}$ defined by Eq.(3.18). Thus the averaged vorticity field decays as $\propto a^{-2}_{\mathcal{D}_p}$ in the expanding universe, which is analogous to the Newtonian linearized theory [22]. In other words, even if the initial global averaged value $\left\langle\mathring{\omega}_i\right\rangle_{\mathring{\mathcal{D}}_p}$ is not zero, we can safely ignore the global averaged vorticity field.

The COBE microwave background measurement suggests that the power spectrum of the density fluctuation field has a positive slope on large scales, supporting the assumption of large-scale homogeneity. This suggests that even if we do not employ the periodicity over the horizon scale, the flux of $\boldsymbol{\mathcal{Q}}$ and $\tilde{\boldsymbol{\mathcal{J}}}$ in Eq.(3.15) through the boundary of the averaging domain with a sufficiently large volume may be negligible. Thus we may conclude that the backreaction on the global expansion rate becomes zero and the equations of the background model introduced by the spatial average of an inhomogeneous universe over the horizon scale obey Friedmann's laws (3.2) and (3.3) in Newtonian cosmology, even when the universe has local nonlinear structures ($\delta\gg 1$) on some small scales.

We may proceed to solve the set of equations (2.4) and (3.14) perturbatively by taking the solutions of the Friedmann's laws as the 0-th order approximation for the scale factor to construct locally inhomogeneous universe [2,4]. This approach is certainly useful, but it is not easy to see the effect of large-scale structure on the small-scale nonlinear dynamics. To see this explicitly we will take a new approach in the next section.

4. HYBRID LAGRANGIAN THEORY FOR NEWTONIAN GRAVITATIONAL INSTABILITY

Since Zel'dovich [27] found the so-called Zel'dovich approximation to describe the large-scale structure formation, the Lagrangian perturbation theory in the cosmological situation have been developed by Buchert [2–4,6] and many other authors [1,9,24]. It has been shown that Zel'dovich approximation gives indeed a very good approximation and has been used up to the quasi non-linear regime to reproduce the observed filament pattern of the large-scale structure beyond several megaparsecs.

The numerical simulation based on the approximation has an advantage over the N-body simulation such that it is able to simulate relatively large domain with relatively small memories. However, the Zel'dovich approximation also has a disadvantage. Namely, it cannot reproduce the non-linear structure formation on small scales once the shell crossing occurs. This is called the *shell crossing problem*. To avoid this, the *truncated* or *optimizing* Lagrangian perturbation approaches have been developed by many authors [9,18,19,26] where one smoothes out the initial small-scale fluctuations such that the shell-crossings occur at about the present time. This will be a good approximation for the evolution of the large-scale structure if one can show that the evolution does not very much depend upon the

behavior of small-scale nonlinear dynamics. However, this expectation has not yet explicitly proven or, put it another way, it has not yet been clarified under what sort of situations this expectation is valid. If this is proved, there may be some hope to have a hybrid way to describing inhomogeneous universe such that the large-scale structure is described by Zel'dovich approximation or some improved version of it and the local small-scale dynamics is described by other method such as N-body simulation or some effective theory of nonlinear structure formation.

This is what we wish to develop in this section. We do this by dividing the deviation field into two parts, the averaged large-scale part and the small-scale part. In the following, we consider the Einstein-de Sitter universe ($\Lambda = \mathcal{K} = 0$) for the sake of simplicity, and the scale factor is normalized as

$$\mathring{a} = 1. \tag{4.1}$$

Namely, we set the scale factor so that the comoving coordinates agree with the Lagrangian coordinates.

A. The division of the deviation field into the large-scale part and the small-scale part

Let us start writing the trajectory field as

$$f_i(\boldsymbol{X},t) \equiv f_i^H(\boldsymbol{X},t) + P_i(\boldsymbol{X},t;\lambda) = a(t)X_i + P_i(\boldsymbol{X},t;\lambda), \tag{4.2}$$

where we have explicitly included the wavelength (λ) dependence in the deviation field P_i. Note that λ here denotes the wavelength of the initial density fluctuation field in the comoving coordinates as explained later. Here the quantity $a(t)$ is the scale factor defined in the previous section: $a = a_{\mathcal{D}_p}$, (where we have omitted the subscript of the quantity $a_{\mathcal{D}_p}$ for simplicity). Namely, the scale factor is obtained by averaging of the Lagrange-Newton system over the horizon scale and obeys Friedmann's law, so the deviation field P_i obeys the periodic boundary condition on the horizon scale:

$$\left\langle P_{i|j}(\boldsymbol{X},t;l<\lambda<L_H)\right\rangle_{\mathring{V}_H} = 0. \tag{4.3}$$

The wavelength of the initial density fluctuations has the lower cutoff because we deal with the collisionless gravitational system like dark matter. Below the lower cutoff this description is not valid since the baryonic gaseous pressure and the effective pressure due to the velocity dispersion of the collisionless system become important. We will not consider such small scales. We have also the upper cutoff which will be the horizon scale L_H because we assumed that the deviation field P_i obeys the periodic boundary condition (4.3) on the horizon scale L_H. We will not explicitly write down these cutoff lengths hereafter.

By substituting the ansatz (4.2) into Eq.(2.5), we can obtain Eq.(3.14), or more explicitly as

$$\begin{aligned} &3a^2\ddot{a} + a^2\ddot{P}_{i|i} + 2a\ddot{a}P_{i|i} + a\left(P_{i|i}\ddot{P}_{j|j} - P_{i|j}\ddot{P}_{j|i}\right) \\ &\quad + \frac{\ddot{a}}{2}\left(P_{i|i}P_{j|j} - P_{i|j}P_{j|i}\right) + \frac{1}{2}\epsilon_{ijk}\epsilon_{abc}P_{i|a}P_{j|b}\ddot{P}_{k|c} = -4\pi G\,\mathring{\rho}\,(\boldsymbol{X};l<\lambda<L_H). \end{aligned} \tag{4.4}$$

Now we introduce the averaged vector field $p_i^>$ by using the spatial average of the full deviation field P_i over the large-scale domain $\mathring{\mathcal{D}}_L(\boldsymbol{X})$ at some point $\boldsymbol{X}$ in the Lagrangian coordinates:

$$\begin{aligned} p_i^>(\boldsymbol{X},t;\lambda \gtrsim L) &\equiv \frac{1}{\mathring{V}_L}\int_{\mathring{\mathcal{D}}_L(X)} d^3X' P_i(\boldsymbol{X}',t;\lambda) \\ &:= \frac{1}{\mathring{V}_L}\int d^3X' P_i(\boldsymbol{X}',t;\lambda)W(\boldsymbol{X},\boldsymbol{X}';L), \end{aligned} \tag{4.5}$$

where L is an artificial cutoff length and

$$\mathring{V}_L := \int_{\mathring{\mathcal{D}}_L(X)} d^3X = \int d^3X' W(\boldsymbol{X},\boldsymbol{X}';L), \tag{4.6}$$

and $W(\boldsymbol{X},\boldsymbol{X}';L)$ is a filter function characterized by the smoothing length L. From Eq.(3.6) the initial condition is imposed on the $p_i^>$:

$$p_i^>(\boldsymbol{X},t_I) = 0. \tag{4.7}$$

We have assumed that, since the smoothing method with the scale L is used for the definition of $p_i^>$, its valid wavelength range has about the length L as a lower cutoff length. Thus, we expect that the cutoff scale length L is large enough so that the $p_i^>$ describes the evolution of fluctuations with the characteristic length larger than L in the linear regime still at the present time t_0, that is, $|p^>_{i|j}(\boldsymbol{X},t_0)| \ll a(t_0)$ is satisfied in the Lagrangian picture. It should be noted that the linear regime in the Lagrangian picture does not mean $\delta \ll 1$ because the density field can be exactly solved. In this situation, we can safely say that the cutoff length L is supposed to be in the range $l \ll L \ll L_H$ in the realistic universe. In the following, we consider the only lowest order in $p_i^>$.

Simultaneously, since the original displacement vector P_i describes the gravitational instability of fluctuations with scale larger than l and smaller than L_H, the definition (4.5) of $p_i^>$ allows us to define the small-scale displacement vector $p_i^<$ which is supposed to describe non-linear structure formations on small scales. Namely, the full displacement vector P_i can be written as

$$P_i(\boldsymbol{X},t;\lambda) \equiv p_i^>(\boldsymbol{X},t;L \lesssim \lambda) + p_i^<(\boldsymbol{X},t;\lambda \lesssim L). \tag{4.8}$$

The initial condition for $p_i^<$ is

$$p_i^<(\boldsymbol{X},t_I) = 0. \tag{4.9}$$

We expect that $p_i^<$ describes the evolution of the initial density fluctuation field with much smaller length than our introducing cutoff length L.

We shall next derive the evolution equations for the large-scale displacement vector $p_i^>$ and the small-scale displacement vector $p_i^<$ obey, respectively. Before doing this we present a basic equation which is obtained by inserting the ansatz (4.8) into Eq.(4.4).

$$\begin{aligned}
&3a^2\ddot{a} + a^2\ddot{p}^>_{i|i} + a^2\ddot{p}^<_{i|i} + 2a\ddot{a}p^>_{i|i} + 2a\ddot{p^<}_{i|i} + a\left(p^>_{i|i}\ddot{p}^<_{j|j} - p^>_{i|j}\ddot{p}^<_{j|i}\right) + a\left(\ddot{p}^>_{i|i}p^<_{j|j} - \ddot{p}^>_{i|j}p^<_{j|i}\right) \\
&+\ddot{a}\left(p^>_{i|i}p^<_{j|j} - p^>_{i|j}p^<_{j|i}\right) + p^>_{i|i}\left[p^<_j\ddot{p}^<_{k|k} - p^<_k\ddot{p}^<_{j|k}\right]_{|j} + p^>_{i|j}\left[2p^<_{j|k}\ddot{p}^<_{k|i} - p^<_{k|k}\ddot{p}^<_{j|i} - \ddot{p}^<_{k|k}p^<_{j|i}\right] \\
&+\frac{1}{2}\ddot{p}^>_{i|i}\left[p^<_jp^<_{k|k} - p^<_kp^<_{j|k}\right]_{|j} + \ddot{p}^>_{i|j}\left[\left(p^<_jp^<_{k|i}\right)_{|k} - \left(p^<_{k|k}p^<_j\right)_{|i}\right] \\
&+a\left(p^<_i\ddot{p}^<_{j|j} - p^<_j\ddot{p}^<_{i|j}\right)_{|i} + \frac{\ddot{a}}{2}\left(p^<_ip^<_{j|j} - p^<_jp^<_{i|j}\right)_{|i} + \frac{1}{2}\left[\ddot{p}^<_i\left(p^<_{j|j}p^<_{k|k} - p^<_{j|k}p^<_{k|j}\right) + 2\ddot{p}_j\left(p^<_{i|k}p^<_{k|j} - p^<_{i|j}p^<_{k|k}\right)\right]_{|i} \\
&+O\left((p^>)^2\right) = -4\pi G\,\mathring{\rho}\,(\boldsymbol{X};l<\lambda<L_H).
\end{aligned} \tag{4.10}$$

In the above derivation, for example, we have used the calculations such as

$$p^<_{i|i}p^<_{j|j} - p^<_{i|j}p^<_{j|i} = \left[p^<_ip^<_{j|j} - p^<_jp^<_{i|j}\right]_{|i}.$$

Note that we have kept only linear order in $p_i^>$ and full order in $p_i^<$. It should be also noted that only the third term in the second line on the left-hand-side of Eq.(4.10) cannot be expressed in the from of a divergence of the vector which consists of $p_i^<$.

B. The evolution equation for the large-scale structure formation

First, we consider the evolution equation for the averaged large-scale field $p_i^>$. Since we have assumed that the $p_i^<$ describes the behavior of fluctuations with the characteristic scale much smaller than L, we introduce the following spatial averaging method on the scale L' much smaller than L and much larger than l: $l \ll L' \ll L$. Namely, the averaging is defined by

$$(1) \qquad \langle g\rangle_{L'}(\boldsymbol{X},t) \equiv \frac{1}{\mathring{V}_{L'}}\int_{\mathring{\mathcal{D}}_{L'}(X)} d^3X'g(\boldsymbol{X}',t) := \frac{1}{\mathring{V}_{L'}}\int d^3X'g(\boldsymbol{X}',t)W(\boldsymbol{X},\boldsymbol{X}';L'). \tag{4.11}$$

Furthermore, within the averaging volume $\mathring{V}_{L'}$, we can safely neglect the spatial gradient of $p_i^>$ because from Eq.(4.5) the $p_i^>$ is defined by the averaging of the volume $\mathring{V}_L$ with $L \gg L'$. Namely, we can safely employ the following second rule when we perform the averaging of an arbitrary function $F(p^<(\boldsymbol{X},t),t)$ of small-scale fluctuations field $p^<$ multiplied by $p^>$ over the volume $\mathring{V}_{L'}$:

(2) $$\frac{1}{\mathring{V}_{L'}}\int_{\mathcal{D}_{L'(X)}} d^3X' p^>(\boldsymbol{X}',t;L<\lambda)F(p^<(\boldsymbol{X}',t),t) = p^>(\boldsymbol{X},t;L<\lambda)\frac{1}{\mathring{V}_{L'}}\int_{\mathcal{D}_{L'(X)}} d^3X' F(p^<(\boldsymbol{X}',t),t). \tag{4.12}$$

This rule is correct at the lowest order of Taylor expansion of $p^>$:

$$p_i^>(\boldsymbol{X}',t) = p_i^>(\boldsymbol{X},t) + p_{i|j}^>(\boldsymbol{X},t)\,(X'-X)_i + \cdots, \tag{4.13}$$

because we have neglected the second term in the above compared with the first term. In the averaging volume $\mathring{V}_{L'}$, the first and second terms is of order of magnitude $p^>$ and $(p^>L')/L$, respectively, so the assumption $L' \ll L$ allows us to ignore the second term compared with the first term. Finally, we introduce the third rule for an arbitrary vector $G_i(p^<)$ which consists of the small-scale displacement vector $p_i^<$:

(3) $$\frac{1}{\mathring{V}_{L'}}\int_{\mathcal{D}_{L'(X)}} d^3X' G_{i|i}(p^<) := \frac{1}{\mathring{V}_{L'}}\int_{\partial\mathcal{D}_{L'(X)}} d^3S_i'\, G_i(p^<) = 0. \tag{4.14}$$

Here, we have assumed that the $p^<$ is mainly generated by initial density fluctuations with only the characteristic wavelength much smaller than L', and obeys the periodic boundary condition on the volume $\mathring{V}_{L'}$ ($\ll\mathring{V}_L$):

$$\left\langle p_{i|j}^<\right\rangle_{L'} = \frac{1}{\mathring{V}_{L'}}\int_{\mathcal{D}_{L'(X)}} d^3X' p_{i|j}^<(\boldsymbol{X}',t) = 0. \tag{4.15}$$

This third rule causes a possible error because we neglect the fluctuations with scale $L' \lesssim \lambda \lesssim L$. Neglecting the fluctuations with the length comparable with the scale L may not be so serious problem because we have assumed that the fluctuations with the scale L is still in the linear regime at the present time. But neglecting the fluctuations with scales comparable with L' might cause a serious problem. For the present we leave this problem open, and we consider the situation under the above three rules.

According to these rules, by averaging the both side of Eq.(4.10) over the domain $\mathring{\mathcal{D}}_{L'}$ we obtain

$$3a^2\ddot{a}(t) + a^2\ddot{p}_{i|i}^>(\boldsymbol{X},t) + 2a\ddot{a}p_{i|i}^> + p_{i|j}^>(\boldsymbol{X},t)\left\langle p_{j|i}^<\ddot{p}_{k|k}^< - \ddot{p}_{j|i}^<p_{k|k}^<\right\rangle_{L'}(\boldsymbol{X},t) = -4\pi G\left\langle\mathring{\rho}\right\rangle_{L'}(\boldsymbol{X};L'\lesssim\lambda), \tag{4.16}$$

where we have used the following calculation

$$\begin{aligned}\left\langle 2p_{j|k}^<\ddot{p}_{k|i}^< - p_{j|i}^<\ddot{p}_{k|k}^< - \ddot{p}_{j|i}^<p_{k|k}^<\right\rangle_{L'} &= \left\langle 2\left(p_j^<\ddot{p}_{k|i}^<\right)_{|k} - 2p_j^<\ddot{p}_{k|ik}^< - p_{j|i}^<\ddot{p}_{k|k}^< - \ddot{p}_{j|i}^<p_{k|k}^<\right\rangle_{L'}\\ &= \left\langle -2\left(p_j^<\ddot{p}_{k|k}^<\right)_{|i} + 2p_{j|i}^<\ddot{p}_{k|k}^< - p_{j|i}^<\ddot{p}_{k|k}^< - \ddot{p}_{j|i}^<p_{k|k}^<\right\rangle_{L'}\\ &= \left\langle p_{j|i}^<\ddot{p}_{k|k}^< - \ddot{p}_{j|i}^<p_{k|k}^<\right\rangle_{L'}(\boldsymbol{X},t).\end{aligned} \tag{4.17}$$

Using the Friedmann equation (3.2), Eq.(4.16) becomes

$$a^2\ddot{p}_{i|i}^>(\boldsymbol{X},t) + 2a\ddot{a}p_{i|i}^>(\boldsymbol{X},t) + p_{i|j}^>(\boldsymbol{X},t)\left\langle p_{j|i}^<\ddot{p}_{k|k}^< - \ddot{p}_{j|i}^<p_{k|k}^<\right\rangle_{L'}(\boldsymbol{X},t) = -4\pi G\left(\left\langle\mathring{\rho}\right\rangle_{L'}(\boldsymbol{X};L'\lesssim\lambda) - \mathring{\rho}_b\right). \tag{4.18}$$

We emphasize that the source function on the right-hand-side of the above equation became the density fluctuation fields with the length larger than the averaging scale L' and smaller than L_H because of using the smoothing method. Furthermore, the third term on the left-hand-side of Eq.(4.18) represents a backreaction effect that the small-scale non-linear displacement vector $p^<$ has on the large-scale perturbation $p^>$. Namely, even if the small non-linear displacement vector $p_i^<$ obeys the periodic boundary condition on the volume $\mathring{V}_{L'}$, the non-linear structures have a possibility to give such a backreaction effect on the evolution of fluctuations with larger scales in the linear regime.

However, we have following situations where the large-scale backreaction becomes zero or negligible compared with the other terms.

- (1) The first case is that the small-scale displacement vector $p_i^<$ can be divided into a time function and a spatial function with respect to the Lagrangian coordinates: $p_i^< = D(t)\psi_i^<$. Then the backreaction term becomes

$$\left\langle p_{j|i}^<\ddot{p}_{k|k}^< - \ddot{p}_{j|i}^<p_{k|k}^<\right\rangle_{L'} = \ddot{D}(t)\left\langle\psi_{j|i}^<\psi_{k|k}^< - \psi_{j|i}^<\psi_{k|k}^<\right\rangle_{L'} = 0.$$

During the early stage after the decoupling time of matter and radiation, we expect that the $p_i^<$ can be described by the Zel'dovich type solution and thus we may have the above case. Actually, Buchert & Ehlers [3] have shown that the displacement vector at every order has such a separable solution in the Lagrangian perturbation theory. But in the non-linear regime later, there is no guarantee that $p_i^<$ is separable. In this paper, we are interested in the non-linear situation of $p_i^<$.

- (2) The second case is as follows. We can divide the deformation field of the large-scale displacement vector $p^>_{i|j}$ into the expansion, shear, and rotation parts without loss of generality:

$$p^>_{i|j} = \frac{1}{3}\delta_{ij}p^>_{k|k} + \left(\frac{1}{2}(p^>_{i|j}+p^>_{j|i}) - \frac{1}{3}\delta_{ij}p^>_{k|k}\right) + \frac{1}{2}(p^>_{i|j}-p^>_{j|i}) \equiv \Theta\delta_{ij} + \Sigma_{ij} + \Omega_{ij}, \tag{4.19}$$

where

$$\Theta(\boldsymbol{X},t) \equiv \frac{1}{3}p^>_{i|i}, \tag{4.20}$$

$$\Sigma_{ij}(\boldsymbol{X},t) \equiv \frac{1}{2}(p^>_{i|j}+p^>_{j|i}) - \frac{1}{3}\delta_{ij}p^>_{k|k}, \tag{4.21}$$

$$\Omega_{ij}(\boldsymbol{X},t) \equiv \frac{1}{2}(p^>_{i|j}-p^>_{j|i}). \tag{4.22}$$

As discussed in appendix A, we can safely ignore the rotation deformation Ω_{ij} compared with the other quantities Θ and Σ under an appropriate assumption: $\Omega_{ij} \approx 0$. Then, if the expansion deformation Θ is much larger than the shear part Σ, the backreaction term becomes

$$p^>_{i|j}\left\langle p^<_{j|i}\ddot{p}^<_{k|k} - \ddot{p}^<_{j|i}p^<_{k|k}\right\rangle_{L'} = \Theta\delta_{ij}\left\langle p^<_{j|i}\ddot{p}^<_{k|k} - \ddot{p}^<_{j|i}p^<_{k|k}\right\rangle_{L'} + \Sigma_{ij}\left\langle p^<_{j|i}\ddot{p}^<_{k|k} - \ddot{p}^<_{j|i}p^<_{k|k}\right\rangle_{L'} = O\left(\langle \Sigma p^< p^<\rangle\right). \tag{4.23}$$

Hence, to ignore the backreaction term compared with the other terms in Eq.(4.18), we have to employ the following condition

$$a^2\ddot{\Theta}, a\ddot{a}\Theta \gg \Sigma_{ij}\left\langle p^<_{j|i}\ddot{p}^<_{k|k} - \ddot{p}^<_{j|i}p^<_{k|k}\right\rangle_{L'}. \tag{4.24}$$

It should be noted that, since $p^>$ represents the averaged large-scale field of the original displacement vector P_i, the assumption $\Theta \gg \Sigma$ seems to be appropriate up to the quasi non-linear regime. The condition (4.24) could be checked in the realistic structure formation.

- (3)The third case is that a locally one-dimensional motion or a spherical motion dominate for $p^<_{i|j}$ in the nonlinear structure formation on small scales:

$$p^<_{i|j} \approx p^<_{1|1}\delta_{i1}\delta_{j1}, \qquad \text{or} \qquad p^<_{i|j} \approx p^<_{k|k}\delta_{ij}. \tag{4.25}$$

In these cases, the backreaction term becomes

$$\begin{aligned} p^>_{i|j}\left\langle p^<_{j|i}\ddot{p}^<_{k|k} - \ddot{p}^<_{j|i}p^<_{k|k}\right\rangle_L &\approx \sum_{\text{clump}} p^>_{i|j}\left\langle p^<_{j|i}\ddot{p}^<_{k|k} - \ddot{p}^<_{j|i}p^<_{k|k}\right\rangle_L \\ &= \sum_{\text{clump}} p^>_{1|1}\left\langle p^<_{1|1}\ddot{p}^<_{1|1} - \ddot{p}^<_{1|1}p^<_{1|1}\right\rangle_L \qquad \text{or} \qquad \sum_{\text{clump}} p^>_{l|l}\delta_{ij}\left\langle p^<_{j|i}\ddot{p}^<_{k|k} - \ddot{p}^<_{j|i}p^<_{k|k}\right\rangle_L \\ &= 0, \end{aligned} \tag{4.26}$$

where $\sum_{\text{clump}}$ denotes the sum of the number of non-linear small-scale clumps included into the averaging domain $\mathring{\mathcal{D}}_L$. Here we have assumed that we can replace the integral in $\langle\cdots\rangle_L$ with the sum over the clumps.

- (4) Finally, we remark that the backreaction term can be rewritten as

$$p^>_{i|j}\left\langle p^<_{j|i}\ddot{p}^<_{k|k} - \ddot{p}^<_{j|i}p^<_{k|k}\right\rangle_{L'} = p^>_{i|j}\frac{d}{dt}\left\langle p^<_{j|i}\dot{p}^<_{k|k} - \dot{p}^<_{j|i}p^<_{k|k}\right\rangle_{L'}. \tag{4.27}$$

This suggests that we may employ a time averaging together with the spatial average to obtain the evolution equation of the large-scale dynamics. Once the nonlinear structures are developed, its time scale will be much shorter than that of large-scale dynamics. Thus, the averaging over the local time scale will eliminate the the above backreaction term. Although this is interesting, we will not pursue it here and leave it for future study in this paper.

Under these situations, the evolution equation (4.18) for the large-scale displacement vector $p_i^>$ yields

$$a^2\ddot{p}^>_{i|i} + 2a\ddot{a}p^>_{i|i} + O\left((p^>)^2\right) = -4\pi G\,\mathring{\rho}_b\mathring{\delta}_{L'}(\boldsymbol{X}; L' \lesssim \lambda), \tag{4.28}$$

where

$$\mathring{\delta}_{L'}(\boldsymbol{X}; L' \lesssim \lambda) \equiv \frac{\left\langle\mathring{\rho}\right\rangle_{L'}(\boldsymbol{X}; L' \lesssim \lambda) - \mathring{\rho}_b}{\mathring{\rho}_b}. \tag{4.29}$$

We again note that, since $p_i^>$ is defined by smoothing the original displacement vector P_i, we need only the lowest order of $p^>$.

Eq.(4.28) entirely agrees with the first order evolution equation in the Lagrangian perturbation theory, so we can proceed to solve it for $p_i^>$ iteratively using the solution of the scale factor $a(t)$ for the Friedmann background equation (3.2). In this way, introducing the artificial cutoff length L and the spatial averaging over the volume $\mathring{V}_{L'}$ smaller than the horizon scale allow us to understand the validity of the Lagrangian perturbation theory in investigating the large-scale structure formation even if the universe has non-linear structures on small scales. Our formalism clarifies the meaning of the optimizing or truncated Lagrangian perturbation theory.

C. The local small-scale non-linear evolution equations in the Lagrangian picture

Next we derive the local evolution equation for the small-scale displacement vector $p_i^<$. We have obtained the background equation and the evolution equation for the large-scale displacement vector within the Lagrangian framework. We want to construct the local evolution equation for the small-scale displacement vector including the effect of the gravitational instability of the surrounding large-scale structure. If we do so, we may have the possibility to use the high-resolution N-body simulation only for the small-scale non-linear structure formation and to use Zel'dovich approximation for the large-scale structure formation, simultaneously.

Subtracting Eq.(4.18) from Eq.(4.10), we obtain the following local small-scale evolution equation in the situations taken up in the previous subsection:

$$\begin{aligned}
&a^2\ddot{p}^<_{i|i} + 2a\ddot{a}p^<_{i|i} + a\left(p^>_{i|i}\ddot{p}^<_{j|j} - p^>_{i|j}\ddot{p}^<_{j|i}\right) + a\left(\ddot{p}^>_{i|i}p^<_{j|j} - \ddot{p}^>_{i|j}p^<_{j|i}\right) + a\left(p^<_{i|i}\ddot{p}^<_{j|j} - p^<_{i|j}\ddot{p}^<_{j|i}\right) + \frac{\ddot{a}}{2}\left(p^<_{i|i}p^<_{j|j} - p^<_{i|j}p^<_{j|i}\right)\\
&\quad + \ddot{a}\left(p^>_{i|i}p^<_{j|j} - p^>_{i|j}p^<_{j|i}\right) + p^>_{i|i}\left(p^<_{j|j}\ddot{p}^<_{k|k} - p^<_{k|j}\ddot{p}^<_{j|k}\right) + p^>_{i|j}\left(2p^<_{j|k}\ddot{p}^<_{k|i} - p^<_{k|k}\ddot{p}^<_{j|i} - \ddot{p}^<_{k|k}p^<_{j|i}\right)\\
&\quad + \frac{1}{2}\ddot{p}^>_{i|i}\left(p^<_{j|j}p^<_{k|k} - p^<_{k|j}p^<_{j|k}\right) + \ddot{p}^>_{i|j}\left(p^<_{j|k}p^<_{k|i} - p^<_{k|k}p^<_{j|i}\right)\\
&\quad + \frac{1}{2}\epsilon_{ijk}\epsilon_{abc}p^<_{i|a}p^<_{j|b}\ddot{p}^<_{k|c} = -4\pi G\left\langle\mathring{\rho}\right\rangle_{L'}\mathring{\delta}\,(\boldsymbol{X}; \lambda \lesssim L'),
\end{aligned} \tag{4.30}$$

where

$$\mathring{\delta}\,(\boldsymbol{X}; \lambda \lesssim L') \equiv \frac{\mathring{\rho}\,(\boldsymbol{X}; \lambda) - \left\langle\mathring{\rho}\right\rangle_{L'}(\boldsymbol{X}; L' \lesssim \lambda)}{\left\langle\mathring{\rho}\right\rangle_{L'}(\boldsymbol{X}; L' \lesssim \lambda)}. \tag{4.31}$$

Note that the source function is a density fluctuations field with scale smaller than L'.

If we could solve Eq.(4.28) for the $p_i^>$, we could solve Eq.(4.30) for the local displacement vector $p_i^<$ in principle by substituting the solution $p^>$. However, in practice it will be very difficult to do so because Eq.(4.30) is highly a nonlinear differential equation. Instead we restrict ourself to more simple situation in this paper. Namely we consider the second situation described in the previous subsection. In this case the expansion dominates on the averaged large-scale dynamics. Then we can rewritten Eq.(4.30) as

$$(a+\Theta)^2\,\ddot{p}^{<}_{i|i}+2(a+\Theta)(\ddot{a}+\ddot{\Theta})p^{<}_{i|i}+(a+\Theta)\left(p^{<}_{i|i}\ddot{p}^{<}_{j|j}-p^{<}_{i|j}\ddot{p}^{<}_{j|i}\right)$$
$$+\frac{1}{2}(\ddot{a}+\ddot{\Theta})\left(p^{<}_{i|i}p^{<}_{j|j}-p^{<}_{i|j}p^{<}_{j|i}\right)+\frac{1}{2}\epsilon_{ijk}\epsilon_{abc}p^{<}_{i|a}p^{<}_{j|b}\ddot{p}^{<}_{k|c}=-4\pi G\left\langle\mathring{\rho}\right\rangle_{L'}\mathring{\delta}(\boldsymbol{X};\lambda\lesssim L'). \tag{4.32}$$

Or using Eqs.(3.18), (4.28) and (4.29), this equation becomes

$$3\,(a+\Theta)^2\left(\ddot{a}+\ddot{\Theta}\right)+(a+\Theta)^2\,\ddot{p}^{<}_{i|i}+2(a+\Theta)(\ddot{a}+\ddot{\Theta})p^{<}_{i|i}+(a+\Theta)\left(p^{<}_{i|i}\ddot{p}^{<}_{j|j}-p^{<}_{i|j}\ddot{p}^{<}_{j|i}\right)$$
$$+\frac{1}{2}(\ddot{a}+\ddot{\Theta})\left(p^{<}_{i|i}p^{<}_{j|j}-p^{<}_{i|j}p^{<}_{j|i}\right)+\frac{1}{2}\epsilon_{ijk}\epsilon_{abc}p^{<}_{i|a}p^{<}_{j|b}\ddot{p}^{<}_{k|c}=-4\pi G\,\mathring{\rho}(\boldsymbol{X};\lambda). \tag{4.33}$$

Note that we have normalized the global scale factor as $\mathring{a}=1$ in this section and considered the Einstein-de Sitter background. If the above equation is compared with the original equation (4.4) in the Lagrangian picture, it is seen that the effect of the large-scale structure formation on the small non-linear scales is represented as a modification of the global scale factor $(a+\Theta)(t)$ on the surrounding large scale.

Thus, since the spatial gradient of the expansion Θ with respect to the Lagrangian coordinates can be ignored inside the volume $\mathring{V}_{L'}$ in this case, the above equation corresponds to the following system of Eulerian equations on the comoving coordinates.

$$\frac{\partial\rho}{\partial t}+3\frac{\dot{a}+\dot{\Theta}}{a+\Theta}\rho+\frac{\partial}{\partial y^i}(\rho v^i)=0, \tag{4.34}$$

$$\frac{\partial v^i}{\partial t}+2\frac{\dot{a}+\dot{\Theta}}{a+\Theta}v^i+v^j\frac{\partial v^i}{\partial y^j}=-\frac{1}{(a+\Theta)^2}\frac{\partial\phi}{\partial y^i}, \tag{4.35}$$

$$\Delta_y\phi=4\pi G(a+\Theta)^2\rho_b\delta, \tag{4.36}$$

where v^i denotes the peculiar velocity on the comoving coordinates $\{\boldsymbol{y}\}$ and the physical peculiar velocity is av^i. The comoving coordinates $\{\boldsymbol{y}\}$ should be defined on the Eulerian coordinates at the initial time, when fluctuations with all scales are much smaller than unity. This equation means that, because of the fact $\Theta<0$ within the large-scale object which are collapsing, the fluctuations on smaller scales tends to collapse earlier than the fluctuations in the large-scale void where has $\Theta>0$. Furthermore, we can perform the N-body simulation using above the Eulerian set of equations inside the box with volume $\mathring{V}_{L'}$ as an usual way only if the scale factor is modified to the effective scale factor $a+\Theta$ inside the box by using the Zel'dovich solution or the improved version of it for $\Theta(t)$. In this way, our formalism will be useful for the consideration of the circumstance effects on the behavior of the fluctuations.

5. DISCUSSION

We have developed the formalism which allows us to investigate the relation between large-scale quasi-linear dynamics and small-scale nonlinear dynamics within using the averaging method in the Lagrangian perturbation theory. We have derived the coupled equation for the large-scale dynamics and the small-scale dynamics. In the case where the averaged large-scale dynamics is expansion dominated, we have shown that the large-scale dynamics decouples with the small-scale nonlinear dynamics. On the other hand, the small-scale dynamics is influenced by the large-scale dynamics in such a way the local small-scale equations contain the modified scale factor in the large scale. The modified scale factor is the sum of the global scale factor and the expansion of the region. Our result strongly suggests that there will be more complicated circumstance effect in local small-scale dynamics which one cannot ignore.

There may be several possibilities to generalize our analysis. One would be to employ some approximation to solve the local dynamics in the Lagrangian framework. For example the spherical symmetric approximation in the small-scale may be reasonable and we may employ Press-Schechter type approach to the local region [17,20,21]. More challenging would be to employ N-body simulation in small scale. We have mentioned this possibility in the previous section, but it seems more work is necessary to achieve this consistently within our scheme. One thing missing in our formalism is the mutual gravitational interaction between the small clumps contained in a large scale. This is because we have ignored the dynamical freedom between $L'<\lambda<L$. We would like to come back this point in future study.

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APPENDIX A: THE LARGE-SCALE TRANSVERSE MODE IN THE LAGRANGIAN PICTURE

In this appendix, we investigate the behavior of the rotational deformation field of the large-scale displacement vector $p_i^>$. For this purpose, it is convenient to make use of the *Helmholtz-Kelvin vorticity transport equation* (2.9) in the Lagrangian picture:

$$\frac{1}{2}\epsilon_{blc}f_{j|b}\dot{f}_{j|c} = \mathring{\omega}_l \ . \tag{A1}$$

By substituting the full trajectory field $f_i = aX_i + p_i^> + p_i^<$ into the above equation, we obtain

$$\frac{1}{2}\epsilon_{ijk}\left[a(\dot{p}^>_{j|k}+\dot{p}^<_{j|k}) - \dot{a}(p^>_{j|k}+p^<_{j|k}) + p^>_{l|j}\dot{p}^<_{l|k} + \dot{p}^>_{l|k}p^<_{l|j} + p^<_{l|j}\dot{p}^<_{l|k}\right] = -\ \mathring{\omega}_i \ (\boldsymbol{X}; l < \lambda < L_H). \tag{A2}$$

If we perform the averaging of the above equation over the domain $\mathring{\mathcal{D}}_{L'}$ according to the rules (4.11), (4.12) and (4.14), we get

$$\frac{1}{2}\epsilon_{ijk}\frac{d}{dt}\left(\frac{p^>_{j|k}}{a}\right) = -\frac{1}{a^2}\left\langle\mathring{\omega}_i\right\rangle_{L'} (\boldsymbol{X}; L' \lesssim \lambda < L_H). \tag{A3}$$

In the above derivation, for example, we have used the calculation such as

$$\frac{1}{2}\epsilon_{ijk}\left\langle p^>_{l|j}\dot{p}^<_{l|k}\right\rangle_{L'} = \frac{1}{2}\epsilon_{ijk}\left\langle (p^>_{l|j}\dot{p}^<_{l})_{|k}\right\rangle_{L'} = 0. \tag{A4}$$

By noting $p^>_{i|j}(\boldsymbol{X}, t_I) = 0$ and using the variable Ω_{ij} defined by Eq.(4.22), Eq.(A3) can be integrated as

$$\frac{1}{2}\epsilon_{ijk}\Omega_{jk} = -a\left\langle\mathring{\omega}_i\right\rangle_{L'}\int_{t_I}^{t} dt\frac{1}{a^2(t')}. \tag{A5}$$

Thus, if we assume that the averaged initial large-scale vorticity field $\langle\omega_i\rangle_{L'}$ is exactly zero, we arrive at the conclusion

$$\Omega_{ij} = 0. \tag{A6}$$

According to this consideration, even if we do not adopt the above assumption, we can safely ignore the large-scale rotational field Ω_{ij} compared with the expansion deformation field Θ and shear field Σ_{ij}, because the initial vorticity field is much smaller than the expansion field and the shear field based on the linearized theory [22].

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