Riemannian geometric aspects of Penrose-type inequalities

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Known Mass Inequalities

The Positive Mass Theorem (Schoen-Yau 1979, Witten 1981)

Among all time-symmetric asymptotically flat initial data sets for the Einstein-Vacuum Equations, flat Euclidean 3-space is the unique minimizer of the total mass.

$$m \geq 0$$

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The Penrose Inequality (Huisken-Ilmanen 2001, Bray 2001)

Among all time-symmetric asymptotically flat initial data sets for the Einstein-Vacuum Equations with an outermost minimal surface Σ of area A, the Schwarzschild slice is the unique minimizer of the total mass.

$$m \geq \frac{1}{2}R \quad (R = \sqrt{A/4\pi})$$

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Natural Questions

The Angular Momentum Case

Is the Kerr slice the unique minimizer of the total mass among all asymptotically flat axisymmetric maximal gauge initial data sets for the Einstein-Vacuum Equations with an outermost minimal surface Σ of area A and (Komar) angular momentum J?

$$m \geq \frac{1}{2}\left(R^2 + \frac{4J^2}{R^2}\right)^{1/2}$$

Natural Questions

The Charged Case I

Is the Reissner-Nordström slice the unique minimizer of the total mass among all asymptotically flat time-symmetric initial data sets for the Einstein-Maxwell Equations with an outermost minimal surface Σ of area A and charge Q?

$$m \geq \frac{1}{2}\left(R + \frac{Q^2}{R}\right) \qquad \left(Q = \int_{S^2_\infty} E \cdot n\right)$$

Recall;

The Time-Symmetric Einstein-Maxwell Contraints

$$S_g = 2(|E|_g^2 + |B|_g^2), \qquad \mathrm{div}_g\, E = \mathrm{div}_g\, B = 0, \qquad E \times_g B = 0$$

Natural Questions

The Charged Case II

(Gibbons '84) Is the Majumdar-Papapetrou slice with the horizon consisting of two components of opposite charges the unique minimizer of the total mass among all asymptotically flat time-symmetric initial data sets for the Einstein-Maxwell Equations with an outermost minimal surface $\Sigma = \cup \Sigma_i$ of area $A = \Sigma A_i$ and charges $\{Q_i\}$?

$$m \geq \sum_i \frac{1}{2}\left(R_i + \frac{Q_i^2}{R_i}\right)$$

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Charged Case I

Answers:

- Yes, provided Σ is connected
  (Jang 1979, Huisken-Ilmanen 2001).

Recall;

The Topology of Horizon Σ

Σ is diffeomorphic to $\cup S^2$
each S^2 stable minimal surface in (M^3, g).

Charged Case I

Answers:

- Yes, provided Σ is connected
  (Jang 1979, Huisken-Ilmanen 2001).
- No, in general
  (Weinstein-Y. 2004).

Recall;

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Charged Case II

Answers: No : Brill-Lindquist satisfies $m < \frac{1}{2}\sum_{i=1}^{2} R_i$ (Dain-Weinstein-Y. 2010).

Remark: By setting quasi-local mass-like quantities;

$$m_i = \frac{1}{2}\left(R_i + \frac{Q_i^2}{R_i}\right), \quad \text{or} \quad \sqrt{\frac{1}{4}R_i^2 + \frac{J_i^2}{R_i^2}}$$

the inequalities

$$m \geq \sum m_i.$$

do not hold, as they reduces to $m \geq \frac{1}{2}\sum R_i$ in vacuum.

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$$\frac{dm_H(\Sigma(t))}{dt} \geq \frac{R}{32\pi} \int_{\Sigma(t)} S_g \geq 0$$

of the Hawking Mass $m_H(\Sigma) = \frac{R}{2}(1 - \frac{1}{16\pi} \int_\Sigma H^2)$ and the scalar curvature is $S_g = 2(|E|^2 + |B|^2)$.

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- Jump over discontinuities with Huisken-Ilamen's weak flow.

Penrose Inequality with Charge when Σ is connected

$$m \geq \frac{1}{2}\left(R + \frac{Q^2}{R}\right) \text{ with '=' iff Reissner-Nordström.}$$

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- The total charge is $Q = 2\mu$.

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An Almost Counter-Example

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- MP is not asymptotically flat.
- MP has no horizon.
- In Weinstein-Y.(2004), two copies of Majumdar-Papapetrouare truncated at their necks, and glued to rectify the features.

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Cosmic Censorship is Safe

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- Our counter-example violates the lower bound.

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A Counter-Example

(Dain-Weinstein-Y. 2010)

$$m - \frac{1}{2}\sum_i R_i < 0$$

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Multiple Components

Stability of $\Sigma \Rightarrow$ multiple components

- If the outermost horizon is connected, then there exists $x \in \Sigma \cap \{z = 0\}$. Then an estimate by Schoen says there exists $\varepsilon > 0$ such that

$$\sup_{\Sigma_1 \cap B(x,\varepsilon)} |A| \leq C \int_{\Sigma_1 \cap B(x,2\varepsilon)} |A|^2 dx$$

  where the RHS is $o(\mu)$ as $\mu \to 0$ and $\Sigma_1 = \Sigma \cap B(x, r)$ with $r < 1$. This says $|\Sigma_1| > C$, with C independent of μ

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- Bray's Penrose inequality

$$2\mu \geq \sqrt{\frac{|\Sigma|}{16\pi}}$$

  says $|\Sigma| \to 0$ as $\mu \to 0$.

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Conformal Flows via Linear Elliptic Equations

Consider the one-parameter familes of metric on $\mathbf{R}^3 \setminus \{0\}$

$$g(t) = \mathcal{U}(t)^4 (dr^2 + r^2 d\omega^2)$$

where

$$\mathcal{U}(t) = e^{-t} + \frac{m}{2re^{-t}}$$

are Schwarzschild metrics, while

$$\mathcal{U}(t) = \left(e^{-t} + \frac{m+Q}{2re^{-t}}\right)^{1/2} \left(e^{-t} + \frac{m-Q}{2re^{-t}}\right)^{1/2}$$

are Reissner-Nordström metrics.

Conformal Flows via Linear Elliptic Equations

Set $u(t) = \mathcal{U}(t)/\mathcal{U}(0)$, then a function $v(t)$ defined by $u(t) = \exp \int_0^t v(\tau) d\tau$ satisfy, for all t,

$$L_{g(t)} v(t) = 0, \quad L_{g(t)} v(t) = \frac{3}{4} |E|_g^2 v(t)$$

for Schwarzschild and Reissner-Nordström metrics respectively, where $L_g = \triangle_g - \frac{1}{8} S_g$. This is crucial to Bray's proof of RPI.

Question:

Any physical meaning to this gauge invariance?

Partial Answer:

Ohashi-Shiromizu-Yamada (2009)

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Results For Solutions of the Einstein-Maxwell Constraints:

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- Σ not connected $\Rightarrow m - \frac{1}{2}\sum_i \left(R_i + \frac{Q_i^2}{R_i}\right) < 0$ is possible (Brill-Lindquist initial data.)

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Open Problems:

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- The Rotating Case
  - $m \geq \frac{1}{2}\left(R^2 + \frac{4J^2}{R^2}\right)^{1/2}$ for Σ connected
  - $\frac{R^2}{2} \leq m^2 + \sqrt{m^4 - J^2}$

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