Exact solutions of Einstein gravity with a negative cosmological constant coupled to a massless scalar field in arbitrary spacetime dimension

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Introduction

- In this work we consider a real massless scalar field minimally coupled to gravity (including a cosmological constant) in arbitrary spacetime dimensions *d*.
- The action is given by

$$I[g_{\mu\nu},\phi] = \int d^dx\sqrt{-g}\left(\frac{R-2\Lambda}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right),$$

- The general static and spherically symmetric solution in four dimensions with $\Lambda = 0$ has a long history, which starts with the Fisher paper in 1948. The generalization in arbitrary higher dimensions was found by Xanthopoulos and Zannias in 1989.

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Main objective

- The goal is to generalize previous results by:

1. including a cosmological constant term, and
2. replacing the $d-2$ sphere by an $(d-2)$-dimensional Einstein manifold Σ, whose Ricci tensor is given by $R_{\Sigma}{}^{m}{}_{n} = (d-3)\gamma\delta^{m}{}_{n}$. The constant γ can be taken to be either 0, $+1$ or -1, depending on whether the intrinsic geometry of the base manifold is flat, spherical, or hyperbolic, respectively.

- Ansatz

$$ds^2 = -e^{2h(r)} f^2(r) dt^2 + \frac{dr^2}{f^2(r)} + r^2 \gamma_{mn} dz^m dz^n, \quad \phi = \phi(r)$$

where γ_{mn} is the metric of Σ.

Field equations

The field equations read

$$R^{\mu}{}_{\nu} - \frac{2\Lambda}{d-2}\delta^{\mu}{}_{\nu} = \kappa\partial^{\mu}\phi\partial_{\nu}\phi, \tag{1}$$

$$\Box\phi = 0. \tag{2}$$

Then, we have the following system of equations for the functions $h(r)$, $f^2(r)$ and $\phi(r)$:

$$\begin{aligned} &(d-3)(\gamma - f^2) - r(h'f^2 + (f^2)') = \frac{2\Lambda}{d-2}r^2, \\ &h' = \frac{\kappa}{(d-2)}r(\phi')^2, \\ &\phi' = \frac{c_0}{e^h f^2 r^{d-2}}. \end{aligned} \tag{3}$$

Here $'$ denotes d/dr, and c_0 is an arbitrary constant that comes from a first integration of (2).

Solving the equations

- Defining the new variable (Das, Gegenberg, Husain, PRD, 2001) $a(r) := r^{d-3}e^h f^2$, one finds from (3)

$$a^2\left[r\frac{a''}{a'} - \frac{2\Lambda r^2 - (d-3)(d-4)\gamma}{\frac{2\Lambda}{d-2}r^2 - (d-3)\gamma}\right] = \frac{\kappa c_0^2}{(d-2)}. \qquad (4)$$

- There are four cases:

1. $\Lambda = 0, \gamma \neq 0 \rightarrow a^2\left[r\frac{a''}{a'} - (d-4)\right] = \frac{\kappa c_0^2}{(d-2)}$
2. $\Lambda \neq 0, \gamma = 0 \rightarrow a^2\left[r\frac{a''}{a'} - (d-2)\right] = \frac{\kappa c_0^2}{(d-2)}$
3. $\Lambda = 0, \gamma = 0 \rightarrow a' = 0$
4. $\Lambda \neq 0, \gamma \neq 0 \rightarrow (4)$.

- We find the exact solution for the first three cases. For the last one there is no an exact solution available . However, it is possible to find an asymptotic solution (r large).

Solving the equations

- We will discuss here the case 2: $\gamma = 0$ with $\Lambda < 0$. The analysis of the remaining cases and more details will be reported soon in arXiv.

$\Lambda < 0, \gamma = 0$

- We find
$$\left(\frac{r}{l}\right)^{d-1} = (a - a_1)^{\frac{a_1}{a_1+a_2}}(a + a_2)^{\frac{a_2}{a_1+a_2}},$$
where $a_1 > 0, a_2 > 0$ are integration constants and $\Lambda = -(d-1)(d-2)/(2l^2)$
- Now, defining the variable $x := a + (a_2 - a_1)/2$, and the constants $b := (a_1 + a_2)/2$ and $p := (a_2 - a_1)/(a_1 + a_2)$, we obtain
$$ds^2 = -(x-b)^{\frac{1+(d-2)p}{(d-1)}}(x+b)^{\frac{1-(d-2)p}{(d-1)}}dt^2 + \frac{l^2}{(d-1)^2}\frac{dx^2}{(x^2-b^2)} + l^2(x-b)^{\frac{1-p}{(d-1)}}(x+b)^{\frac{1+p}{(d-1)}}\gamma_{mn}dz^m dz^n,$$
$$\phi(x) = \phi_0 + \sqrt{\frac{d-2}{d-1}}\sqrt{\frac{1-p^2}{4\kappa}}\ln\left(\frac{x-b}{x+b}\right).$$
The solution contains three integration constants b, p, and ϕ_0. Note that $|p| \leq 1$, $b > 0$ and $x \geq b$.

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$\Lambda < 0, \gamma = 0$

- The Ricci scalar reads

$$R = -\frac{(d-1)}{l^2}\left[d-(d-2)\frac{b^2(1-p^2)}{(x^2-b^2)}\right].$$

- In general, for a non trivial ϕ, i.e. $b \neq 0$ and $p^2 \neq 1$, there is a curvature singularity at $x = b$, which corresponds to $r = 0$. This singularity is a naked singularity at the origin.
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Asymptotic behavior

We now turn to study the asymptotic behavior. We find that for $\gamma \neq 0$

$$a(r) = \left(\frac{r}{l}\right)^{d-1} + \gamma \left(\frac{r}{l}\right)^{d-3} - \mu + O\left(r^{-(d-1)}\right), \tag{5}$$

where μ is an arbitrary constant. In terms of the integration constants of the above exact solution for the case $\gamma = 0$, the constant $\mu = 2bp$. With the result of eq. (5) we obtain the asymptotic expansion of the metric:

$$\begin{aligned} ds^2 = & -\left[\left(\frac{r}{l}\right)^2 + \gamma - \mu\left(\frac{l}{r}\right)^{d-3} + O\left(r^{-2(d-2)}\right)\right] dt^2 \\ & + \left[\left(\frac{r}{l}\right)^2 + \gamma - \mu\left(\frac{l}{r}\right)^{d-3} + O\left(r^{-2(d-2)}\right)\right]^{-1} dr^2 + r^2 \gamma_{mn} dz^m dz^n. \end{aligned}$$

We note that the spacetime is a locally asymptotically AdS spacetime.

Asymptotic behavior

- The asymptotic form of the scalar field is given by

$$\phi = \phi_0 - \phi_1 \left(\frac{l}{r}\right)^{d-1} + O\left(r^{-(d+1)}\right),$$

  where ϕ_0 and ϕ_1 are arbitrary constants.
- The family of asymptotic solutions is thus parametrized by the three constants μ, ϕ_0, and ϕ_1.

Mass

- Starting from the asymptotic behavior of the metric and the scalar field, we now turn to the problem of computing the mass of these configurations. We address this issue following the Regge-Teitelboim approach.
- In general, for the model considered here, the variation of the conserved charges corresponding to the asymptotic symmetries defined by the vector $\xi = (\xi^t, \xi^i)$, is given by

$$\delta Q(\xi) = \delta Q_G(\xi) + \delta Q_\phi(\xi), \text{with}$$

$$\begin{aligned}\delta Q_G(\xi) &= \frac{1}{2\kappa}\int d^{d-2}S_l G^{ijkl}(\xi^\perp \delta g_{ij;k} - \xi^\perp{}_{,k}\delta g_{ij}) \\ &+ \int d^{d-2}S_l(2\xi_k\delta\pi^{kl} + (2\xi^k\pi^{jl} - \xi^l\pi^{jk})\delta g_{jk})\end{aligned}$$

$$\delta Q_\phi(\xi) = -\int d^{d-2}S_l(\xi^\perp g^{1/2}g^{lj}\partial_j\phi\delta\phi + \xi^l\pi_\phi\delta\phi).$$

Mass

- Here g_{ij} denotes the components of the $(d-1)$-spatial metric, π^{ij} are their conjugate momenta, π_ϕ is the momentum associated to ϕ. We have also defined $\xi^\perp = \xi^t\sqrt{-g_{tt}}$, and

$$G^{ijkl} \equiv \frac{1}{2}g^{1/2}(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}).$$

In the static case all the momenta vanish, and the relevant asymptotic symmetry corresponds to the vector ∂_t. We then write the variation of the mass as $\delta M = \delta Q(\partial_t) = \delta M_G + \delta M_\phi$. We obtain

$$\delta M_G = -\lim_{r\to\infty}\frac{(d-2)}{2\kappa}V(\Sigma)\frac{r^{d-2}}{l}(g^{rr})^{-1/2}\delta g^{rr} = \frac{(d-2)}{2\kappa}V(\Sigma)l^{d-3}\delta\mu,$$

$$\delta M_\phi = -\lim_{r\to\infty}V(\Sigma)\frac{r^{d-1}}{l}(g^{rr})^{1/2}\phi'\delta\phi = (d-1)V(\Sigma)l^{d-3}\phi_1\delta\phi_0,$$

where $V(\Sigma)$ denotes the volume of the Einstein base manifold.

Mass

- The next step is to integrate the variations δM_G and δM_ϕ in order to obtain the value of M. The gravitational contribution can be directly integrated, giving the result

$$M_G = \frac{(d-2)}{2\kappa} V(\Sigma) l^{d-3} \mu.$$

- For the scalar field contribution the problem is more subtle, since δM_ϕ depends on the product $\phi_1 \delta\phi_0$. This variation only can be integrated if:

A) ϕ_0 and ϕ_1 are related as $\phi_1 = \phi_1(\phi_0)$ or vice versa.

B) $\delta\phi_0 = 0$.

- In the case (A) $M_\phi \neq 0$ and the scalar field contributes to the mass, whose value depends on the relation $\phi_1 = \phi_1(\phi_0)$. This occurs for massive scalar fields with slow fall off in AdS spaces.

Mass

- Case (B) is possible when ϕ_0 vanishes or it is a constant without variation. Symmetry conditions could require this. Indeed, there is an asymptotic scale invariance in the field equations.
- An infinitesimal scaling σ produces a variation in ϕ as $\delta_\sigma \phi = r\sigma\phi' \sim \phi_1 / r^{d-1}$. This variation is compatible with the functional variation of the scalar field $\delta\phi = \delta\phi_0 - \delta\phi_1 / r^{d-1}$ only if $\delta\phi_0 = 0$
- Thus, we conclude that under this symmetry condition, only the "gravitational" piece contributes to the mass and this is given by $M = M_G$.