Nonlinear perturbations in the GR limit of Hořava-Lifshitz gravity

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Introduction

- Inspired by Lifshitz scalars in condensed matter physics, Hořava introduced a power-counting renormalizable gravity theory. Hořava 2009
- High (spatial) curvature terms ⇒ UV improvement.
- Only two time derivatives are allowed, avoiding ghosts.
- The anisotropic scaling implies violation of Lorentz invariance. The theory should recover Lorentzian symmetry in IR.

HL gravity has many interesting cosmological applications

- Bouncing/Oscillating universe – singularity avoidance *(See poster by Y. Misonoh)*
  Calcagni'09; Brandenberger'09; Wang, Wu'09; Misonoh, Maeda, Kobayashi'10/'11; ...
- Solution to the horizon problem without inflation Mukohyama'09
- Milder flatness problem Kiritsis, Kofinas'09
- Circular polarized primordial GW Takahashi, Soda'09
- Generation of primordial magnetic fields Maeda, Mukohyama, Shiromizu'09
- ...

However, there are a few issues related to an extra degree of freedom. In this talk, I will concentrate in one of these issues, the "strong coupling problem".

Construction of the action for $z = 3$

- Introducing an anisotropic scaling
$$\left.\begin{array}{ccc} x & \to & b^{-1}\,x \\ t & \to & b^{-z}\,t \end{array}\right\} \begin{array}{c} d+1 \text{ dimensional QFT with } \partial_t^2,\ \partial_i^{2z} \\ \text{power} - \text{counting renormalizable for } z \geq d \end{array}$$
- Symmetry: foliation-preserving diffeomorphism
$$t \to t'(t) \qquad \vec{x} \to \vec{x}'(t, \vec{x})$$
- ADM decomposition provides a natural parametrization of the field content.
$$ds^2 = -N^2\,dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$
- The allowed ingredients: $N\,dt\,,\sqrt{g}d^3x\,,K_{ij}\,,g_{ij}\,,D_i\,,R_{ij}$,
where $K_{ij} = \frac{1}{2\,N}\left[\partial_t\,g_{ij} - D_i N_j - D_j N_i\right]$.
- Projectability condition: $N = N(t)$, consistent with the F-P diffeomorphism.
- The most general action for $z = 3$ scaling can be written as
$$S = \frac{M_{\rm Pl}^2}{2}\int N\,dt\,\sqrt{g}\,d^3\vec{x}\left(K^{ij}K_{ij} - \lambda K^2 - 2\Lambda + R + L_{z>1}\right),$$
with $$\frac{M_{Pl}^2}{2}L_{z>1} = \left(c_1\,D_iR_{jk}D^iR^{jk} + c_2\,D_iR\,D^iR + c_3R_i^jR_j^kR_k^i + c_4R\,R_i^jR_j^i + c_5R^3\right) + \left(c_6R_i^jR_j^i + c_7R^2\right).$$

IR limit?

- If theory is renormalizable, coupling constants run under RG flow. At low energies, GR can be recovered only if $\lambda \to 1$.

$$S = \frac{M_{\rm Pl}^2}{2} \int N\, dt \sqrt{g}\, d^3\vec{x} \left(K^{ij} K_{ij} - \lambda K^2 - 2\Lambda + R \right),$$

- Caution: Renormalizability of Hořava-Lifshitz gravity is not yet proven. We do not yet know how RG flow proceeds.
- Even if $\lambda = 1$, resulting theory still not GR! Due to projectability $\Rightarrow S \ni \int dt\, N(t) \left(\int d^3x \sqrt{g}\, \mathcal{H} \right)$, i.e. no local Hamiltonian constraint,

$$\int d^3x \sqrt{g}\, \mathcal{H} = 0$$

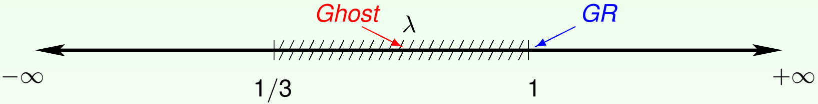
$\Rightarrow$ GR + "Dark matter as an integration constant": Mukohyama 2009.

$$M_p^2\, G^{(4)}_{\mu\nu} = T^{(4)}_{\mu\nu} + \rho_{\text{“}DM\text{''}} n_\mu n_\nu$$

- Degrees of freedom: 2 tensor graviton + 1 scalar graviton.

Issues with scalar graviton

1. $1/3 < \lambda < 1 \Longrightarrow$ Ghost
2. $\lambda < 1/3,\ \lambda > 1 \Longrightarrow c_s^2 = -\frac{\lambda-1}{3\lambda-1} < 0 \Rightarrow$ Instability.
3. $\lambda \to 1 \Longrightarrow$ Strong coupling



- Instability may be hidden by: expansion, Jeans instability of ordinary matter or geometric "dark matter". `Mukohyama 2010.`
- Phenomenologicaly viable range: $1 < \lambda < \infty$.
- Possible weak coupling at $\lambda \to \infty$ `AEG, Mukohyama 2011.`
- Strong coupling $\to$ breakdown of perturbations $\Longrightarrow$ Loss of predictability? : All coefficients of infinite number of interactions can be expressed in terms of the finite parameters in the full action, if the theory is renormalizable.
- For an accurate analysis: need nonperturbative methods.

Breakdown of perturbative expansion?

- Perturbing flat FRW in vacuum:
  $$N = 1, \quad N_i = \partial_i B + n_i, \quad g_{ij} = a^2\, \mathrm{e}^{\zeta_T} \left(\mathrm{e}^{h}\right)_{ij} \;;\; \left[\partial_i h_{ij} = h_{ii} = 0\right]$$
- Momentum constraint, at first order in perturbations:
  $$\partial_i \left[a^2(3\lambda - 1)\partial_t \zeta_T - (\lambda - 1)\triangle B\right] + \tfrac{1}{2}\triangle n_i = 0$$
  solved by:
  $$\triangle B = \tfrac{3\lambda-1}{\lambda-1}\, a^2 \partial_t \zeta_T, \quad n_i = 0$$
- After eliminating B in the action,
  $$S_{\,\mathrm{kin}} \ni \int dt\, d^3x\, a^3 \left[\frac{(\cdots)}{\lambda - 1}\, \dot{\zeta}_T^2 + \frac{(\cdots)}{(\lambda - 1)^2}\, \dot{\zeta}_T^2\, \zeta_T\right]$$
- Perturbative treatment gives a convergent action if $\boxed{|\zeta_T| \ll \min(1, \lambda - 1)}$.
- But, is the solution of mom. cons. valid outside this range?

Momentum constraint: Second branch of solution

AEG, Mukohyama, Wang '11

- Consider small metric perturbations for $\zeta = \mathcal{O}(q),\ h_{ij} = \mathcal{O}(q)$
- But keep the shift vector nonlinear $B = \mathcal{O}(q^0),\ n_i = \mathcal{O}(q^0)$
- Momentum constraint, at leading order

$$\mathcal{H}_i = -(3\lambda - 1)\,\partial_i\partial_t\zeta_T + \mathcal{O}(q^2) - \frac{1}{2\,a^2}\,[\triangle + \mathcal{O}(q)]\,n_i$$
$$+\frac{1}{a^2}\left\{(\lambda-1)\left[\delta^j_i\triangle + \mathcal{O}(q)\right] + \left(\frac{1}{2}\triangle h^j{}_i + \partial^j\partial_i\zeta_T + \delta^j_i\,\triangle\zeta_T + \mathcal{O}(q^2)\right)\right\}\partial_j B\,.$$

Two branches of solutions

1. $\boxed{\text{for } q \ll \min(1,\ \lambda-1)} \Longrightarrow \triangle B = \frac{3\lambda-1}{\lambda-1}\,a^2\,\partial_t\zeta_T + \mathcal{O}(q^2)\,, \qquad n_i = \mathcal{O}(q^2)$
2. $\boxed{\text{for } \lambda - 1 \ll q \ll 1} \Longrightarrow \mathcal{O}(q)$ terms are larger than $\mathcal{O}(q^0)$ terms

$$B = 2\,a^2\left[\tfrac{1}{2}\,\triangle h^i_j + \partial^i\partial_j\zeta_T + \delta^i_j\triangle\zeta_T + 2\,(\partial^i\triangle\zeta_T)\,\partial_i\right]^{-1}\triangle\partial_t\zeta_T + \mathcal{O}\left(\frac{\lambda-1}{q}\right) + \mathcal{O}(q)$$

- Branch 2 is nonlinear regime $\Rightarrow$ but still consistent with $\zeta, h_{ij} \ll 1$.
- For the $\lambda \to 1$ limit, Branch 2 is relevant.
- Nonlinear dynamics take over $\Rightarrow$ Vainshtein mechanism   Vainshtein 1972

Examples of Vainshtein effect analogues

1. Static, spherically symmetric vacuum configurations continuous to GR at $\lambda \to 1$ Mukohyama 2010
2. Approximation:
   Gradient Expansion, $H^{-1} \ll L$ Salopek, Bond 1990; Lyth, Malik, Sasaki 2005
   - Small parameter: $\epsilon \sim 1/HL \Rightarrow \partial_i \sim \mathcal{O}(\epsilon)$.
   - Amplitude of perturbations does not have to be small.
     $\Rightarrow$ At a given order in ϵ, all terms from perturbative expansion contribute.

   Gradient expansion around flat FRW:
   - HL in vacuum – continuous to GR+"DM" Izumi, Mukohyama 2011
   - Extend to include scalar field AEG, Mukohyama, Wang 2011
     $\Rightarrow$ No pathologies as $\lambda \to 1$, recovers GR+"DM"+Scalar.

These studies are consistent with our argument that nonlinear effects removes the divergences.

Summary

- Fully nonperturbative study of superhorizon perturbations in HL gravity+Scalar field, shows no pathological behavior in the limit $\lambda \to 1$. Theory is reduced to GR + "DM" + Scalar $\Rightarrow$ another analogue of Vainshtein effect.
- In contrast, naive application of perturbative expansion gives rise to divergences. We found that these solutions are not valid as $\lambda \to 1$. A new branch of nonlinear solutions found. Regular as $\lambda \to 1$, still consistent with small metric perturbations.
- Gradient expansion: Classical analysis. Quantum analogue? Integrating out nonlinear B, can we get a healthy perturbative action?
- Similar analyses of the extended versions of the theory?

- Renormalizability beyond power-counting not proven yet. RG flow of coupling constant has not been investigated. The conjecture that $\lambda \to 1$ to be IR fixed point is based on our hope to recover GR at low energies.
- How to avoid LV leaking to matter sector?
  Supersymmetrization of SM? `Groot Nibbelink, Pospelov '05`
  Leak may be under control if $M \ll M_p$? `Pospelov, Shang '10`

Extra Slides

Details: Power-counting renormalizability

Example:

- The action for a free Lifshitz scalar in $3+1$, with dynamical critical exponent z:

$$S = \int dt\, d^3x \left[\dot{\phi}^2 - \phi\,(-\triangle)^z\,\phi\right]$$

- Rescaling the momentum by $k \to b\,k$, we have

$$\left.\begin{array}{lcl} x & \to & b^{-1}\,x \\ t & \to & b^{-z}\,t \\ \phi & \to & b^s\,\phi \end{array}\right\} \underbrace{-z}_{dt}\,\underbrace{-3}_{d^3x} + \overbrace{2\,z}^{\partial} + \overbrace{2\,s}^{\phi^2} = 0 \Rightarrow s = \frac{3-z}{2}$$

- For Lorentz-invariant case ($z=1$), $\phi \to b\,\phi$.
- For $z=3$, ϕ is dimensionless.

Details: Power-counting renormalizability - Interactions

- With up to $2z$ spatial derivatives
- Including interactions

$$S_{\text{int}} = \int dt\, d^3x\, g_n \phi^n$$

- Momentum dimension of the coupling constant

$$\underbrace{-z}_{dt}\ \underbrace{-3}_{d^3x} + [g_n] + \frac{\overbrace{n\,(3-z)}^{\phi^n}}{2} = 0 \Longrightarrow [g_n] = 3 + z - \frac{n(3-z)}{2}$$

- For Lorentz-invariant case ($z = 1$) $\Rightarrow [g_n] = 4 - n$
  Renormalizable if $n \leq 4$.
- For $z = 3$, the operators are dimensionless, so $[g_n] = 6$ for any n.
- Gravitational interactions (which are highly nonlinear), exhibit a power counting renormalizability for $z \geq 3$.
- We will set $z = 3$ from now on $\Rightarrow$ include up to 6 spatial derivatives.

Details: "Dark matter" as an integration constant

Assuming $\lambda \to 1$ in the IR,

- Dynamical equations: $G^{(4)}_{ij} + \Lambda g^{(4)}_{ij} - \frac{1}{M_p^2} T_{ij} = 0\,,$
- Momentum constraint: $\left(G^{(4)}_{i\mu} + \Lambda g^{(4)}_{i\mu} - \frac{1}{M_p^2} T_{i\mu}\right) n^\mu = 0\,, \qquad (n_\mu dx^\mu = Ndt)$
- Hamiltonian constraint: $\int d^3x\sqrt{g}\left(G^{(4)}_{\mu\nu} + \Lambda g^{(4)}_{\mu\nu} - \frac{1}{M_p^2} T_{\mu\nu}\right) n^\mu n^\nu = 0\,.$
- Defining $T^{\text{``}DM\text{''}}_{\mu\nu} \equiv M_p^2\left(G^{(4)}_{\mu\nu} + \Lambda g^{(4)}_{\mu\nu} - \frac{1}{M_p^2} T_{\mu\nu}\right) \Rightarrow \begin{array}{l} T^{\text{``}DM\text{''}}_{ij} = 0 \\ T^{\text{``}DM\text{''}}_{i\mu} n^\mu = 0 \\ \Rightarrow T_{\mu\nu} = \rho_{\text{``}DM\text{''}}\, n_\mu n_\nu \end{array}$

- The extra piece in the equations then looks like pressureless dust with energy density and rest frame synchronized with the foliation. Mukohyama 2009
- Hamiltonian constraint: $\int d^3x\sqrt{g}\,\rho_{\text{``}DM\text{''}} = 0 \Rightarrow$ total energy of the dust vanishes. When our horizon is approximated by FRW, the super horizon inhomogeneities in $\rho_{\text{``}DM\text{''}}$ should cancel the homogeneous and positive contribution in our patch.
- Generically, $\rho_{\text{``}DM\text{''}}$ is generated away from $\lambda = 1$

$$n^\mu \nabla_\mu \rho_{\text{``}DM\text{''}} + K\,\rho_{\text{``}DM\text{''}} = n^\mu \nabla^\nu T_{\nu\mu} + \mathcal{O}(\lambda - 1) + \mathcal{O}(R^2)$$

Details: Issues with scalar graviton

Gravitational action, scalar sector

$$S_g^{(2)} = M_p^2 \int dt\, d^3x \left[\left(\frac{3\,\lambda - 1}{\lambda - 1} \right) \dot{\zeta}^2 - \zeta \left(\frac{\triangle}{a^2} + \frac{\kappa_s\, \triangle^2}{a^4 M_s^2} - \frac{\triangle^3}{a^6\, M_s^4} \right) \zeta \right]$$

$$\frac{1}{M_s^4} \equiv -2\, \frac{3\, c_1 + 8\, c_2}{M_{\rm Pl}^2}$$

- For $1/3 < \lambda < 1$, wrong sign kinetic term $\Rightarrow$ Scalar mode is a ghost.
- Instability: Dispersion relation:

$$\omega^2 = \frac{\lambda - 1}{3\,\lambda - 1} \left(\frac{k^6}{a^6\, M_s^4} + \frac{\kappa_s\, k^4}{a^4 M_s^2} - \frac{k^2}{a^2} \right)$$

For $k/a < M_s$, scalar graviton is unstable, with time scale $t_I \sim \frac{a}{k} \sqrt{\left| \frac{3\,\lambda - 1}{\lambda - 1} \right|}$.

However, as in CDM scenario, the dust-like component exhibits Jeans instability, with time scale $t_J = \frac{1}{\sqrt{G\,\rho}}$. The linear instability does not show up if $t_I > t_J$ or $t_I > H^{-1}$.

$$0 < \frac{\lambda - 1}{3\,\lambda - 1} < \max \left(\frac{H^2 a^2}{k^2}\, , |\Phi| \right)\, , \qquad H < \frac{k}{a} < \min \left(M_s\, , \frac{1}{.01\text{mm}} \right)$$

This is a phenomenological constraint on RG flow. λ should approach to 1 (from above) as the energy scale is lowered.

Details: Gradient expansion around flat FRW (summary)

Summary of gradient expansion around flat FRW + scalar field calculation

- Gravitational action: $S_g = \frac{M_{Pl}^2}{2}\int N dt\,\sqrt{g}\,d^3\vec{x}\,\left(K_{ij}K^{ij} - \lambda K^2 - 2\Lambda + R + L_{z>1}\right)$
- Matter action: $S_\phi = \int N\,dt\,\sqrt{g}\,d^3\vec{x}\left[\frac{1}{2}(\partial_\perp\phi)^2 - V_0(\phi) - V_{z\geq 1}(\phi, D_i, g_{ij})\right]$
- Synchronous gauge: $N = 1$, $N_i = 0$, $\partial_\perp \to \partial_t$

$$ds^2 = -dt^2 + a(t)^2\,\mathrm{e}^{2\,\zeta(t,\vec{x})}\,\gamma_{ij}(t,\vec{x})dx^i dx^j\,, \qquad (\det\gamma = 1)$$

- Assumption: Locally FRW $\Rightarrow \partial_t\gamma_{ij} \sim \mathcal{O}(\epsilon)$.
- Additional assumption: $\partial_i\phi \sim \mathcal{O}(\epsilon^2)$.
- Equations of motion can be solved order by order.
- Momentum constraint satisfied at any order.
- Just like in the vacuum result, no pathologies at $\lambda \to 1$.
- Recovers GR+"DM"+Scalar field.

- These studies are consistent with our argument that nonlinear effects removes the divergences.

Details: Gradient expansion around flat FRW (equations)

- In the synchronous gauge ($N = 1,\ N_i = 0$), equations of motion are

$$\begin{aligned}
(3\lambda - 1)\partial_t K &= -\frac{1}{2}(3\lambda - 1)K^2 - \frac{3}{2}A^i{}_j A^j{}_i - \frac{3}{2}(\partial_t\phi)^2 - Z\,, \\
\partial_t A^i{}_j &= -KA^i{}_j + Z^i{}_j - \frac{1}{3}Z\,\delta^i{}_j, \\
0 &= \partial_t^2\phi + K\partial_t\phi + E_\phi\,, \\
\partial_t\zeta &= -\frac{\partial_t a}{a} + \frac{1}{3}K\,, \\
\partial_t\gamma_{ij} &= 2\,\gamma_{ik}A^k{}_j\,,
\end{aligned}$$

$$A^i_j = K^i_j - \tfrac{1}{3}\,\delta^i_j K$$

$$K = 3\,\left(\partial_t\zeta + \tfrac{\partial_t a}{a}\right)$$

- Momentum constraint

$$\partial_j A^j{}_i + 3\,A^j{}_i\partial_j\zeta - \frac{1}{2}\,A^j{}_l(\gamma^{-1})^{lk}\partial_i\gamma_{jk} - \frac{1}{3}\,(3\lambda - 1)\,\partial_i K = \partial_t\phi\,\partial_i\phi\,.$$

- Expansion:

$$\begin{aligned}
\zeta &= \zeta^{(0)}(\vec{x}) + \epsilon\,\zeta^{(1)}(t,\vec{x}) + \epsilon^2\zeta^{(2)}(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,, \\
\gamma_{ij} &= f_{ij}(\vec{x}) + \epsilon\,\gamma^{(1)}_{ij}(t,\vec{x}) + \epsilon^2\gamma^{(2)}_{ij}(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,, \\
K &= 3\,H(t) + \epsilon\,K^{(1)}(t,\vec{x}) + \epsilon^2 K^{(2)}(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,, \\
A^i{}_j &= \epsilon\,A^{(1)\,i}{}_j(t,\vec{x}) + \epsilon^2 A^{(2)\,i}{}_j(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,, \\
\phi &= \phi^{(0)}(t) + \epsilon\,\phi^{(1)}(t,\vec{x}) + \epsilon^2\phi^{(2)}(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,,
\end{aligned}$$

Details: First order momentum constraint → action

- Full kinetic action, quadratic and cubic:

$$S^{(2)}_{kin} = \int dt d^3\vec{x} a^3 \left(a^{-2}\partial_t \zeta_T \triangle B + \frac{1}{8}\partial_t h^{ij}\partial_t h_{ij} \right),$$

$$S^{(3)}_{kin} = \int dt d^3\vec{x} a^3 \Big[3\zeta_T \left(a^{-2}\partial_t \zeta_T \triangle B + \frac{1}{8}\partial_t h^{ij}\partial_t h_{ij} \right) + \frac{1}{2} a^{-4}\zeta_T \partial^i(\partial_i B \triangle B + 3\partial^j B \partial_i \partial_j B)$$
$$+ \frac{1}{2}(a^{-2}\partial^k h^{ij}\partial_k B - 3\partial_t h^{ij}\zeta_T) a^{-2}\partial_i\partial_j B - \frac{1}{4} a^{-2}\partial_t h^{ij}\partial_k h_{ij}\partial^k B \Big].$$

- At first order: $\partial_i \left[a^2(3\lambda - 1)\partial_t \zeta_T - (\lambda - 1)\triangle B \right] + \frac{1}{2}\triangle n_i = 0$
  solved by:

$$\triangle B = \frac{3\lambda - 1}{\lambda - 1} a^2 \partial_t \zeta_T, \quad n_i = 0$$

Relation between synchronous gauge and transverse gauge

$$\zeta = -\frac{2}{3}\frac{1}{\lambda - 1}\Big\{ \zeta_T - \frac{(3\,\lambda - 1)}{(\lambda - 1)} (\partial_i \zeta_T)(\partial^i \triangle^{-1}\zeta_T) + \frac{(3\,\lambda - 1)}{(\lambda - 1)} \int^t dt' \, (\partial_i \zeta_T)\left(\partial^i \triangle^{-1}\partial_{t'}\zeta_T\right)$$
$$- \frac{(3\,\lambda - 1)}{2\,(\lambda - 1)} \int^t dt' \triangle^{-1} \Big[2\,(\partial^i \triangle \zeta_T)(\partial_i \triangle^{-1}\partial_{t'}\zeta_T)$$
$$+ \left(\partial^i \partial^j \zeta_T + \frac{1}{2}\triangle h^{ij} \right)(\partial_i \partial_j \triangle^{-1}\partial_{t'}\zeta_T) + (\triangle \zeta_T)(\partial_{t'}\zeta_T) \Big]$$
$$+ \frac{1}{4}\int^t dt' \, \triangle^{-1}\left[\frac{1}{2}(\partial_i \partial_{t'} h_{jk})(\partial^i h_{jk}) + \frac{1}{2}(\partial_{t'} h^{ij})(\triangle h_{ij}) - 3\,(\partial_i \partial_j \zeta_T)(\partial_{t'} h^{ij}) \right] + \mathcal{O}(\bar{\epsilon}^3) \Big\}$$