Constraint propagation and constraint-damping for C^2-adjusted formulations

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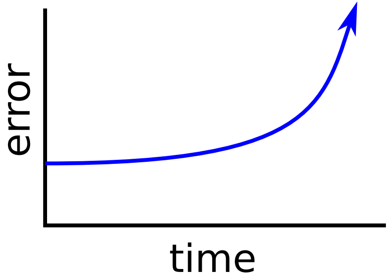
Introduction and Motivation

In numerical relativity, we must set the implementations:

- Initial Condition
- Boundary Condition
- Gauge Condition
- Formulation
- Scheme
- etc.

In this presentation, I will talk about the Formulation.

Introduction and Motivation



The simulations stop since the violations of constraints increase, even if we set all implementations appropriately expect for the formulation.

It is called formulation problem in numerical relativity.

Currently,

- The ADM formulation is not used.
- The Baumgarte-Shapiro-Shibata-Nakamura (BSSN) formulation is widely used.

My purpose is to obtain more stable and robust formulations.

Introduction and Motivation

One of the improvement of formulations is to add the constraint to the evolution equation. (It is called the constraint damping technique.)

adjusted system [1]

If we add the constraint to the evolution equation

$$\partial_t u^i = [\textbf{Original Terms}] + f(C^i, \partial_j C^i, \cdots) \quad (1)$$

Then the constraint propagation equation becomes

$$\partial_t C^i = [\textbf{Original Terms}] + g(C^i, \partial_j C^i, \cdots) \quad (2)$$

With fixing the background in generally, we can predict the behavior of the constraints of the new system from the eigenvalue analysis of (2).

- How we set $f(C^i, \partial_j C^i, \cdots)$ in (1)?
- If the background changes, is the prediction right?

$\Rightarrow$ I will introduce a method of setting $f(C^i, \partial_j C^i, \cdots)$ which is without depending on the background. It is the C^2-adjusted system.

[1] G. Yoneda and H. Shinkai in PRD **63**, 124019 and PRD **66**, 124003

General Idea of C^2-adjusted system

Suppose an evolution system with constraint

$$\begin{cases} \partial_t u^i = f^i(u^i, \partial_j u^i, \dots) \\ C^i = g^i(u^i, \partial_j u^i, \dots) \approx 0 \end{cases} \tag{3}$$

If the evolution equation is adjusted as

$$\partial_t u^i = f^i(u^i, \partial_j u^i, \dots) - \kappa^{ij} \frac{\delta C^2}{\delta u^j} \tag{4}$$

$$\text{where,} \quad C^2 = \int C^i C_i dx^3, \quad \kappa^{ij} : \text{Positive definite} \tag{5}$$

Then, the constraint propagation equation becomes

$$\partial_t C^2 = [\textbf{Original terms}] - \kappa^{ij} \left(\frac{\delta C^2}{\delta u^i}\right)\left(\frac{\delta C^2}{\delta u^j}\right) \tag{6}$$

(This idea is suggested by D. R. Fiske in PRD **69**, 047501 (2004))

General Idea of C^2-adjusted system

Suppose an evolution system with constraint

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Then, the constraint propagation equation becomes

$$\partial_t C^2 = [\textbf{Original terms}] - \kappa^{ij} \left(\frac{\delta C^2}{\delta u^i}\right) \left(\frac{\delta C^2}{\delta u^j}\right) < 0 \tag{6}$$

(This idea is suggested by D. R. Fiske in PRD **69**, 047501 (2004))

Outline

Standard ADM Formulation

The evolution equations:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta(\gamma_{ij}) \tag{7}$$

$$\partial_t K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K_i{}^\ell K_{\ell j}) - \nabla_i \nabla_j \alpha + \mathcal{L}_\beta(K_{ij}) \tag{8}$$

The constraint equations:

$$\mathcal{H}^{ADM} = R + K^2 - K_{ij}K^{ij} \approx 0 \tag{9}$$

$$\mathcal{M}_i^{ADM} = \nabla_j K^j{}_i - \nabla_i K \approx 0 \tag{10}$$

C^2-adjusted ADM Formulation

The evolution equations of the C^2-adjusted ADM formulation:

$$\partial_t \gamma_{ij} = [\textbf{Original Terms}] - \kappa_{\gamma ijmn} \frac{\delta (C^{ADM})^2}{\delta \gamma_{mn}} \tag{11}$$

$$\partial_t K_{ij} = [\textbf{Original Terms}] - \kappa_{Kijmn} \frac{\delta (C^{ADM})^2}{\delta K_{mn}} \tag{12}$$

where

$$(C^{ADM})^2 = \int \{(\mathcal{H}^{ADM})^2 + \gamma^{ij}(\mathcal{M}_i^{ADM})(\mathcal{M}_j^{ADM})\} dx^3 \tag{13}$$

Constraint Propagation Equations

The constraint propagation equations with the C^2-adjusted ADM formulation in flat spacetime and $\kappa_{\gamma ijmn} = \kappa_\gamma \delta_{im}\delta_{jn}, \kappa_{Kijmn} = \kappa_K \delta_{im}\delta_{jn}$:

$$\partial_t \mathcal{H} = [\textbf{Original Terms}] - 2\kappa_\gamma \Delta^2 \mathcal{H} \tag{14}$$

$$\partial_t \mathcal{M}_i = [\textbf{Original Terms}] + \kappa_K \Delta \mathcal{M}_i + 3\kappa_K \partial_j \partial_i \mathcal{M}^j \tag{15}$$

In the additional terms, there are the diffusion terms (red terms).

Outline

Standard BSSN Formulation

The definition of the dynamical variables:

$$\varphi = \frac{1}{12}\log(\det(\gamma_{ij})) \tag{16}$$

$$\tilde{\gamma}_{ij} = e^{-4\varphi}\gamma_{ij} \tag{17}$$

$$K = \gamma^{ij}K_{ij} \tag{18}$$

$$\tilde{A}_{ij} = e^{-4\varphi}\left(K_{ij} - \frac{1}{3}\gamma_{ij}K\right) \tag{19}$$

$$\tilde{\Gamma}^i = \tilde{\gamma}^{ab}\tilde{\Gamma}^i{}_{ab} \tag{20}$$

Standard BSSN Formulation

The evolution equations:

$$\partial_t \varphi = -(1/6)\alpha K + (1/6)(\partial_i \beta^i) + \beta^i(\partial_i \varphi) \tag{21}$$

$$\partial_t K = \alpha \widetilde{A}_{ij}\widetilde{A}^{ij} + (1/3)\alpha K^2 - D_i D^i \alpha + \beta^i(\partial_i K) \tag{22}$$

$$\partial_t \widetilde{\gamma}_{ij} = -2\alpha \widetilde{A}_{ij} - (2/3)\widetilde{\gamma}_{ij}(\partial_\ell \beta^\ell) + \widetilde{\gamma}_{j\ell}(\partial_i \beta^\ell) + \widetilde{\gamma}_{i\ell}(\partial_j \beta^\ell) + \beta^\ell(\partial_\ell \widetilde{\gamma}_{ij}) \tag{23}$$

$$\begin{aligned}\partial_t \widetilde{A}_{ij} &= \alpha K \widetilde{A}_{ij} - 2\alpha \widetilde{A}_{i\ell}\widetilde{A}^\ell{}_j + \alpha e^{-4\varphi} R_{ij}{}^{TF} - e^{-4\varphi}(D_i D_j \alpha)^{TF} \\ &\quad - (2/3)\widetilde{A}_{ij}(\partial_\ell \beta^\ell) + (\partial_i \beta^\ell)\widetilde{A}_{j\ell} + (\partial_j \beta^\ell)\widetilde{A}_{i\ell} + \beta^\ell(\partial_\ell \widetilde{A}_{ij})\end{aligned} \tag{24}$$

$$\begin{aligned}\partial_t \widetilde{\Gamma}^i &= 2\alpha\{6(\partial_j \varphi)\widetilde{A}^{ij} + \widetilde{\Gamma}^i{}_{j\ell}\widetilde{A}^{j\ell} - (2/3)\widetilde{\gamma}^{ij}(\partial_j K)\} - 2(\partial_j \alpha)\widetilde{A}^{ij} \\ &\quad + (2/3)\widetilde{\Gamma}^i(\partial_j \beta^j) + (1/3)\widetilde{\gamma}^{ij}(\partial_\ell \partial_j \beta^\ell) + \beta^\ell(\partial_\ell \widetilde{\Gamma}^i) - \widetilde{\Gamma}^j(\partial_j \beta^i) \\ &\quad + \widetilde{\gamma}^{j\ell}(\partial_j \partial_\ell \beta^i)\end{aligned} \tag{25}$$

Standard BSSN formulation

The constraint equations:
The "kinematic" constraint equations:

$$\mathcal{H}^{BSSN} \equiv e^{-4\varphi}\tilde{R} - 8e^{-4\varphi}(\tilde{D}_i\tilde{D}^i\varphi + (\tilde{D}^m\varphi)(\tilde{D}_m\varphi)) + (2/3)K^2 - \tilde{A}_{ij}\tilde{A}^{ij} - (2/3)\mathcal{A}K \approx 0 \quad (26)$$

$$\mathcal{M}_i^{BSSN} \equiv -(2/3)\tilde{D}_iK + 6(\tilde{D}_j\varphi)\tilde{A}^j{}_i + \tilde{D}_j\tilde{A}^j{}_i - 2(\tilde{D}_i\varphi)\mathcal{A} \approx 0, \quad (27)$$

$$(28)$$

The "algebraic" constraints:

$$\mathcal{G}^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{j\ell}\tilde{\Gamma}^i{}_{j\ell} \approx 0 \quad (29)$$

$$\mathcal{A} \equiv \tilde{A}^{ij}\tilde{\gamma}_{ij} \approx 0 \quad (30)$$

$$\mathcal{S} \equiv \det(\tilde{\gamma}_{ij}) - 1 \approx 0 \quad (31)$$

If the algebraic constraints are not satisfied, the BSSN formulation and ADM formulation are not equivalent mathematically.

C^2-adjusted BSSN Formulation

The evolution equations of the C^2-adjusted BSSN formulation:

$$\partial_t \varphi = [\textbf{Original Terms}] - \lambda_\varphi \left(\frac{\delta (C^{BSSN})^2}{\delta \varphi} \right) \tag{32}$$

$$\partial_t K = [\textbf{Original Terms}] - \lambda_K \left(\frac{\delta (C^{BSSN})^2}{\delta K} \right) \tag{33}$$

$$\partial_t \tilde{\gamma}_{ij} = [\textbf{Original Terms}] - \lambda_{\tilde{\gamma}ijmn} \left(\frac{\delta (C^{BSSN})^2}{\delta \tilde{\gamma}_{mn}} \right) \tag{34}$$

$$\partial_t \tilde{A}_{ij} = [\textbf{Original Terms}] - \lambda_{\tilde{A}ijmn} \left(\frac{\delta (C^{BSSN})^2}{\delta \tilde{A}_{mn}} \right) \tag{35}$$

$$\partial_t \tilde{\Gamma}^i = [\textbf{Original Terms}] - \lambda_{\tilde{\Gamma}}^{ij} \left(\frac{\delta (C^{BSSN})^2}{\delta \tilde{\Gamma}^j} \right) \tag{36}$$

where

$$(C^{BSSN})^2 = \int \{ (\mathcal{H}^{BSSN})^2 + \gamma^{ij} (\mathcal{M}^{BSSN})_i (\mathcal{M}^{BSSN})_j + \gamma_{ij} \mathcal{G}^i \mathcal{G}^j + \mathcal{A}^2 + \mathcal{S}^2 \} dx^3$$

Constraint Propagation Equations

The constraint propagation equations with the C^2-adjusted BSSN formulation in flat spacetime and $\lambda_{\tilde{\gamma}ijmn} = \lambda_{\tilde{\gamma}}\delta_{im}\delta_{jn}$, $\lambda_{\tilde{A}ijmn} = \lambda_{\tilde{A}}\delta_{im}\delta_{jn}$, $\lambda_{\tilde{\Gamma}}^{ij} = \lambda_{\tilde{\Gamma}}\delta^{ij}$:

$$\begin{aligned}
\partial_t \mathcal{H} &= [\textbf{Original Terms}] + \{-128\lambda_\varphi \Delta^2 - (3/2)\lambda_{\tilde{\gamma}}\Delta^2 + 2\lambda_{\tilde{\Gamma}}\Delta\}\mathcal{H} \\
&\quad + \{-(1/2)\lambda_{\tilde{\gamma}}\Delta\partial_m - 2\lambda_{\tilde{\Gamma}}\partial_m\}\mathcal{G}^m + 3\lambda_{\tilde{\gamma}}\Delta\mathcal{S} \qquad (37) \\
\partial_t \mathcal{M}_a &= [\textbf{Original Terms}] + \{(8/9)\lambda_K \delta^{bc}\partial_a\partial_b + \lambda_{\tilde{A}}\Delta\delta_a{}^c + \lambda_{\tilde{A}}\delta^{bc}\partial_a\partial_b\}\mathcal{M}_c \\
&\quad - 2\lambda_{\tilde{A}}\partial_a\mathcal{A} \qquad (38) \\
\partial_t \mathcal{G}^a &= [\textbf{Original Terms}] + \delta^{ab}\{(1/2)\lambda_{\tilde{\gamma}}\partial_b\Delta + 2\lambda_{\tilde{\Gamma}}\partial_b\}\mathcal{H} - \lambda_{\tilde{\gamma}}\delta^{ab}\partial_b\mathcal{S} \\
&\quad + \left(\lambda_{\tilde{\gamma}}\Delta\delta^a{}_b + (1/2)\lambda_{\tilde{\gamma}}\delta^{ac}\partial_c\partial_b - 2\lambda_{\tilde{\Gamma}}\delta^a{}_b\right)\mathcal{G}^b \qquad (39) \\
\partial_t \mathcal{A} &= [\textbf{Original Terms}] + 2\lambda_{\tilde{A}}\delta^{ij}(\partial_i\mathcal{M}_j) - 6\lambda_{\tilde{A}}\mathcal{A} \qquad (40) \\
\partial_t \mathcal{S} &= [\textbf{Original Terms}] + 3\lambda_{\tilde{\gamma}}\Delta\mathcal{H} + \lambda_{\tilde{\gamma}}\partial_\ell\mathcal{G}^\ell - 6\lambda_{\tilde{\gamma}}\mathcal{S} \qquad (41)
\end{aligned}$$

In the additional terms, there are the diffusion terms (red terms).

Constraint Propagation Equations

If the $(C^{BSSN})^2$ do not include the algebraic constraints($\mathcal{G}^i$, $\mathcal{A}$, and $\mathcal{S}$):

$$(C^{BSSN})^2 = \int \{(\mathcal{H}^{BSSN})^2 + \gamma^{ij}(\mathcal{M}^{BSSN})_i(\mathcal{M}^{BSSN})_j\} dx^3,$$

Then, the constraint propagation equations become

$$\partial_t \mathcal{H} = [\textbf{Original Terms}] + \{-128\lambda_\varphi \Delta^2 - (3/2)\lambda_{\tilde{\gamma}}\Delta^2 + 2\lambda_{\tilde{\Gamma}}\Delta\}\mathcal{H} \quad (42)$$

$$\partial_t \mathcal{M}_a = [\textbf{Original Terms}] + \{(8/9)\lambda_K \delta^{bc}\partial_a\partial_b + \lambda_{\tilde{A}}\Delta\delta_a{}^c + \lambda_{\tilde{A}}\delta^{bc}\partial_a\partial_b\}\mathcal{M} \quad (43)$$

$$\partial_t \mathcal{G}^a = [\textbf{Original Terms}] + \delta^{ab}\{(1/2)\lambda_{\tilde{\gamma}}\partial_b\Delta + 2\lambda_{\tilde{\Gamma}}\partial_b\}\mathcal{H} \quad (44)$$

$$\partial_t \mathcal{A} = [\textbf{Original Terms}] + 2\lambda_{\tilde{A}}\delta^{ij}(\partial_i\mathcal{M}_j) \quad (45)$$

$$\partial_t \mathcal{S} = [\textbf{Original Terms}] + 3\lambda_{\tilde{\gamma}}\Delta\mathcal{H} \quad (46)$$

The propagation equations of the algebraic constraints do not include the diffusion terms.
$\Rightarrow$ $(C^{BSSN})^2$ should include the algebraic constraints.

Outline

Test Metric

Polarized Gowdy wave testbed (One of apples-with-apples testbed[2])

$$ds^2 = t^{-1/2} e^{\lambda/2}(-dt^2 + dx^2) + t(e^P dy^2 + e^{-P} dz^2) \quad (47)$$

$$P = J_0(2\pi t)\cos(2\pi x) \quad (48)$$

$$\lambda = -2\pi t J_0(2\pi t) J_1(2\pi t)\cos^2(2\pi x) + 2\pi^2 t^2 [J_0^2(2\pi t) + J_1^2(2\pi t)] - (1/2)\{(2\pi)^2 [J_0^2(2\pi) + J_1^2(2\pi)] - 2\pi J_0(2\pi) J_1(2\pi)\} \quad (49)$$

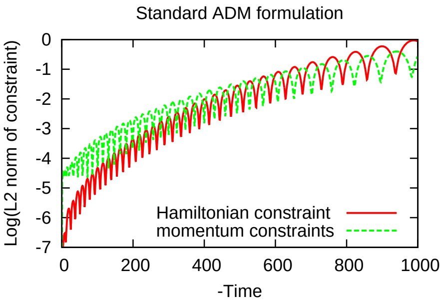
where J_n is the Bessel function.

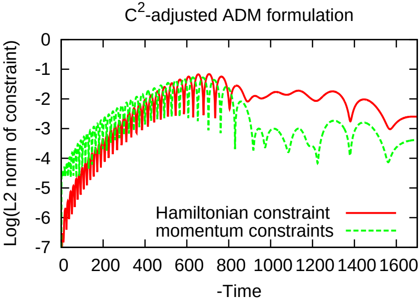
I performed the other tests of apples-with-apples testbeds (gauge-wave and linear-wave testbeds), but I show only the Gowdy wave testbed.

[2] Alcubierre *et al.*, Class. Quant. Grav. **21**, 589 (2004)

Outline

Numerical Test with ADM Formulations



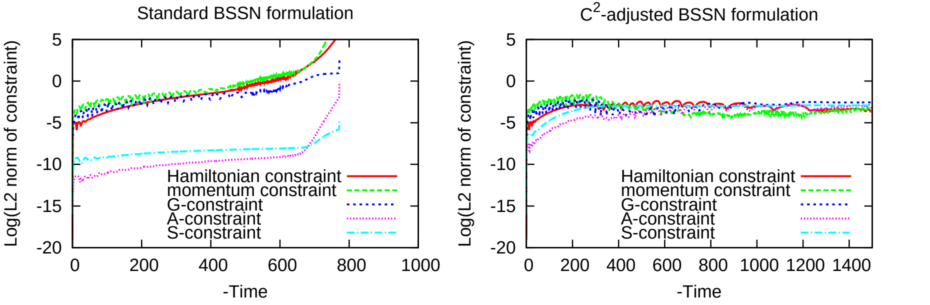


- The lifetime of the simulation of the C^2-adjusted ADM formulation become longer (1.7 times).
- The violations of constraints of the C^2-adjusted ADM formulaiton decrease.

(This result is accepted in Phys. Rev. **D** 83, 064032 (2011))

Outline

Numerical Test with BSSN Formulations



- The lifetime of the simulation of the C^2-adjusted BSSN formulation become longer (2 times).
- The violations of the constraints of the C^2-adjusted BSSN formulation keep.

(This result is submitted to PRD (gr-qc/1109.5782))

Summary and Future Work

Summary

- We apply the C^2-adjusted system to the ADM and BSSN formulations.
- We show the constraint propagation equations with the C^2-adjusted ADM and BSSN formulations. We see that these equations include the damping terms.
- In the C^2-adjusted BSSN formulation, C^2 should include the algebraic constraints from the analysis of the constraint propagation equation.
- We perform some simulations with the C^2-adjusted formulations and the lifetime is longer than each of the standard formulation.

Future Work

- The C^2-adjusted first order ADM formulation.
- The method of the setting of the Lagrange multiplier.