Violation of the Rotational Invariance in the CMB bispectrum

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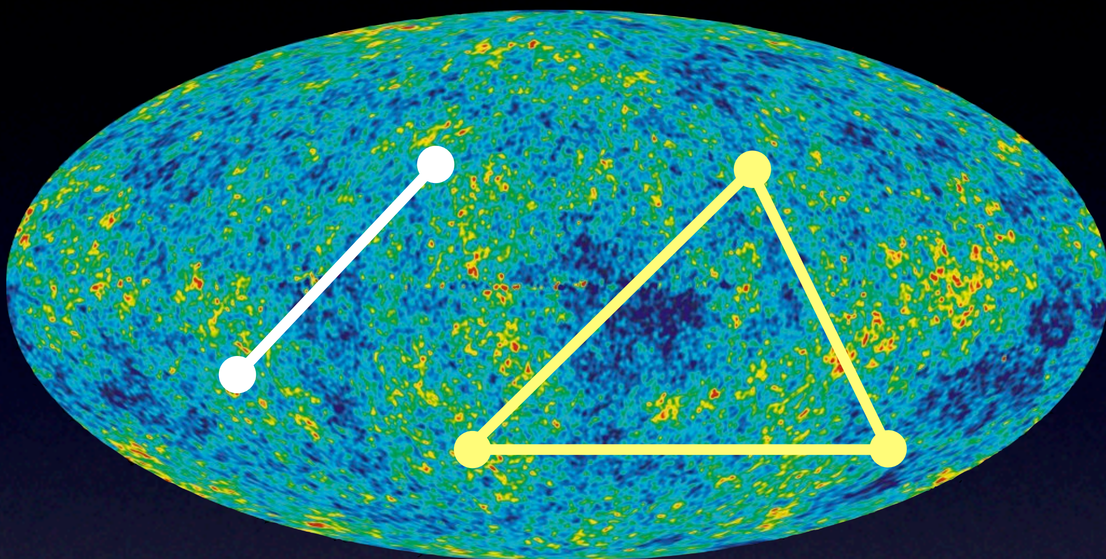
based on arXiv:1107.0682 (accepted in PTP)

CMB Bispectrum



power spectrum: $<\Delta T(n_1)\Delta T(n_2)>$

bispectrum: $<\Delta T(n_1)\Delta T(n_2)\Delta T(n_3)>$

$\propto <\xi(k_1)\xi(k_2)\xi(k_3)>$

$\neq 0$ if ξ is non-Gaussian

$$\langle \Phi_{\mathbf{k}_1}\Phi_{\mathbf{k}_2}\Phi_{\mathbf{k}_3}\rangle = (2\pi)^3\delta^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)F(k_1,k_2,k_3)$$

$$\begin{aligned} &F_{\rm local}(k_1,k_2,k_3) \\ &= 2f_{NL}^{\rm local}[P_\Phi(k_1)P_\Phi(k_2)+P_\Phi(k_2)P_\Phi(k_3) \\ &\quad +P_\Phi(k_3)P_\Phi(k_1)] \\ &= 2A^2 \underline{f_{NL}^{\rm local}}\left[\frac{1}{k_1^{4-n_s}k_2^{4-n_s}} + (2\text{ perm.})\right], \end{aligned}$$

$$-10 \ < \ f_{NL}^{\rm local} \ < \ 74$$

Komatsu + [1001.4538]

In the previous work, the rotational invariance is assumed

What is the Rotational invariance?

power spectrum

k-space:

$$P(\mathbf{k}) = P(k)$$

absence of the angular dependence

CMB fluctuation is expanded with spherical harmonics

l-space:

$$\left\langle \prod_{i=1}^{2} a_{\ell_i m_i} \right\rangle \equiv C_{\ell_1} (-1)^{m_1} \delta_{\ell_1,\ell_2} \delta_{m_1,-m_2}$$

bispectrum

k-space:

$$F^{\lambda_1\lambda_2\lambda_3}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) = f^{\lambda_1\lambda_2\lambda_3}(k_1,k_2,k_3) \times [\text{polarization vectors or tensors}]$$

$$\frac{\Delta X(\hat{n})}{X} = \sum_{\ell m} a_{X,\ell m} Y_{\ell m}(\hat{n})$$

l-space:

$$\left\langle \prod_{i=1}^{3} a_{\ell_i m_i} \right\rangle \equiv \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1 \ell_2 \ell_3}.$$

If the rotational invariance violates, three azimuthal quantum numbers is not confined in this Wigner-3j symbol!!

An Inflation model with a preferred direction

S. Yokoyama & J. Soda [astro-ph: 0805.4265]

System like the hybrid inflation that there are inflaton φ, waterfall field χ, and a vector field A_μ coupled with a waterfall field

$$S = \int dx^4 \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} g^{\mu\nu} (\partial_\mu \phi \partial_\nu \phi + \partial_\mu \chi \partial_\nu \chi) - V(\phi, \chi, A_\nu) - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} f^2(\phi) F_{\mu\rho} F_{\nu\sigma} \right] .$$

Using the δN formalism, the total curvature perturbation of superhorizon limit on uniform energy density hypersurface is given by

$$\begin{aligned} \zeta(t_{\rm e}) &= \delta N(t_{\rm e}, t_*) \\ &= \frac{\partial N}{\partial \phi_*} \delta\phi_* + \frac{1}{2} \frac{\partial^2 N}{\partial \phi_*^2} \delta\phi_*^2 + \frac{\partial N}{\partial \phi_{\rm e}} \frac{d\phi_{\rm e}(A)}{dA^\mu} \delta A^\mu_{\rm e} \\ &\quad + \frac{1}{2} \left[\frac{\partial N}{\partial \phi_{\rm e}} \frac{d^2 \phi_{\rm e}(A)}{dA^\mu dA^\nu} + \frac{\partial^2 N}{\partial \phi_{\rm e}^2} \frac{d\phi_{\rm e}(A)}{dA^\mu} \frac{d\phi_{\rm e}(A)}{dA^\nu} \right] \delta A^\mu_{\rm e} \delta A^\nu_{\rm e} \end{aligned}$$

At the end of inflation, δA generates additional ζ through δφ

An Inflation model with a preferred direction

S.Yokoyama & J. Soda [astro-ph: 0805.4265]

Power spectrum of primordial curvature perturbations:

$$\left\langle \prod_{n=1}^{2} \zeta(\boldsymbol{k_n}) \right\rangle = (2\pi)^3 N_*^2 P_\phi(k_1) \delta\left(\sum_{n=1}^{2} \boldsymbol{k_n}\right) + N_e^2 \frac{d\phi_e(A)}{dA^\mu} \frac{d\phi_e(A)}{dA^\nu} \langle \delta A_e^\mu(\boldsymbol{k_1}) \delta A_e^\nu(\boldsymbol{k_2}) \rangle$$

Set the Coulomb gauge and solve the evolution equation of the vector field for f ∝ a, a^{-2}:

$$A_i'' - \frac{f''}{f} A_i - a^2 \partial_j \partial^j A_i = 0$$

$$\langle \delta A_e^i(\boldsymbol{k_1}) \delta A_e^j(\boldsymbol{k_2}) \rangle = (2\pi)^3 P_\phi(k) f_e^{-2} P^{ij}(\hat{\boldsymbol{k_1}}) \delta\left(\sum_{n=1}^{2} \boldsymbol{k_n}\right)$$

scale-invariant spectrum

$$P_\phi(k) = H_*^2/(2k^3)$$

$$N_* \equiv \partial N/\partial \phi_*$$

$$N_e \equiv \partial N/\partial \phi_e$$

$' \equiv d/d\tau$

$$A_i \equiv f \delta A_i$$

$$P^{ij}(\hat{\boldsymbol{k}}) = \delta^{ij} - \hat{k}^i \hat{k}^j$$

$$f_e \equiv f(t_e)$$

$$q_i \equiv d\phi_e/dA^i$$

$$g_\beta = -\frac{\beta}{1+\beta}$$

$$\beta = (N_e/N_*/f_e)^2 |\boldsymbol{q}|^2$$

$$\left\langle \prod_{n=1}^{2} \zeta(\boldsymbol{k_n}) \right\rangle \equiv (2\pi)^3 P_\zeta(\boldsymbol{k_1}) \delta\left(\sum_{n=1}^{2} \boldsymbol{k_n}\right)$$

$$P_\zeta(\boldsymbol{k}) \equiv P_\zeta^{\mathrm{iso}}(k) \left[1 + g_\beta \left(\hat{\boldsymbol{q}} \cdot \hat{\boldsymbol{k}}\right)^2\right]$$

$$P_\zeta^{\mathrm{iso}}(k) = N_*^2 P_\phi(k)(1+\beta)$$

g_β: the magnitude of anisotropy

An Inflation model with a preferred direction

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Bispectrum of primordial curvature perturbations:

$$\left\langle \prod_{n=1}^{3} \zeta(\boldsymbol{k_n}) \right\rangle = (2\pi)^3 N_*^2 N_{**} [P_\phi(k_1)P_\phi(k_2) + 2 \text{ perms.}]\delta\left(\sum_{n=1}^{3}\boldsymbol{k_n}\right)$$
$$+N_{\rm e}^3 \frac{d\phi_{\rm e}(A)}{dA^\mu}\frac{d\phi_{\rm e}(A)}{dA^\nu}\frac{d\phi_{\rm e}(A)}{dA^\rho}\langle \delta A_{\rm e}^\mu(\boldsymbol{k_1})\delta A_{\rm e}^\nu(\boldsymbol{k_2})\delta A_{\rm e}^\rho(\boldsymbol{k_3})\rangle$$
$$+N_{\rm e}^4 \frac{d\phi_{\rm e}(A)}{dA^\mu}\frac{d\phi_{\rm e}(A)}{dA^\nu}\left(\frac{1}{N_{\rm e}}\frac{d^2\phi_{\rm e}(A)}{dA^\rho dA^\sigma} + \frac{N_{\rm ee}}{N_{\rm e}^2}\frac{d\phi_{\rm e}(A)}{dA^\rho}\frac{d\phi_{\rm e}(A)}{dA^\sigma}\right)$$
$$\times [\langle \delta A_{\rm e}^\mu(\boldsymbol{k_1})\delta A_{\rm e}^\nu(\boldsymbol{k_2})(\delta A^\rho \star \delta A^\sigma)_{\rm e}(\boldsymbol{k_3})\rangle + 2 \text{ perms.}] \ ,$$

$$N_{**} \equiv \partial^2 N/\partial\phi_*^2$$
$$N_{\rm ee} \equiv \partial^2 N/\partial\phi_{\rm e}^2$$
$$\hat{q}^{cd} \equiv q^{cd}/|\boldsymbol{q}|^2$$
$$q_{ij} \equiv d^2\phi_{\rm e}/(dA^i dA^j)$$

assume δA almost obeys Gaussian statistics, hence neglect the cubic term of δA

$$\left\langle \prod_{n=1}^{3} \zeta(\boldsymbol{k_n}) \right\rangle \equiv (2\pi)^3 F_\zeta(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3})\delta\left(\sum_{n=1}^{3}\boldsymbol{k_n}\right)$$

= slow roll parameter ~ O(0.01), so negligible

$$F_\zeta(\boldsymbol{k_1},\boldsymbol{k_2},\boldsymbol{k_3}) = \left(\frac{g_\beta}{\beta}\right)^2 P_\zeta^{\rm iso}(k_1)P_\zeta^{\rm iso}(k_2)\left[\frac{N_{**}}{N_*^2} + \beta^2\hat{q}^a\hat{q}^b\left(\frac{1}{N_{\rm e}}\hat{q}^{cd} + \frac{N_{\rm ee}}{N_{\rm e}^2}\hat{q}^c\hat{q}^d\right)P_{ac}(\hat{\boldsymbol{k_1}})P_{bd}(\hat{\boldsymbol{k_2}})\right] +2 \text{ perms.}$$

This term generates a direction-depending non-Gaussianity!!

An Inflation model with a preferred direction

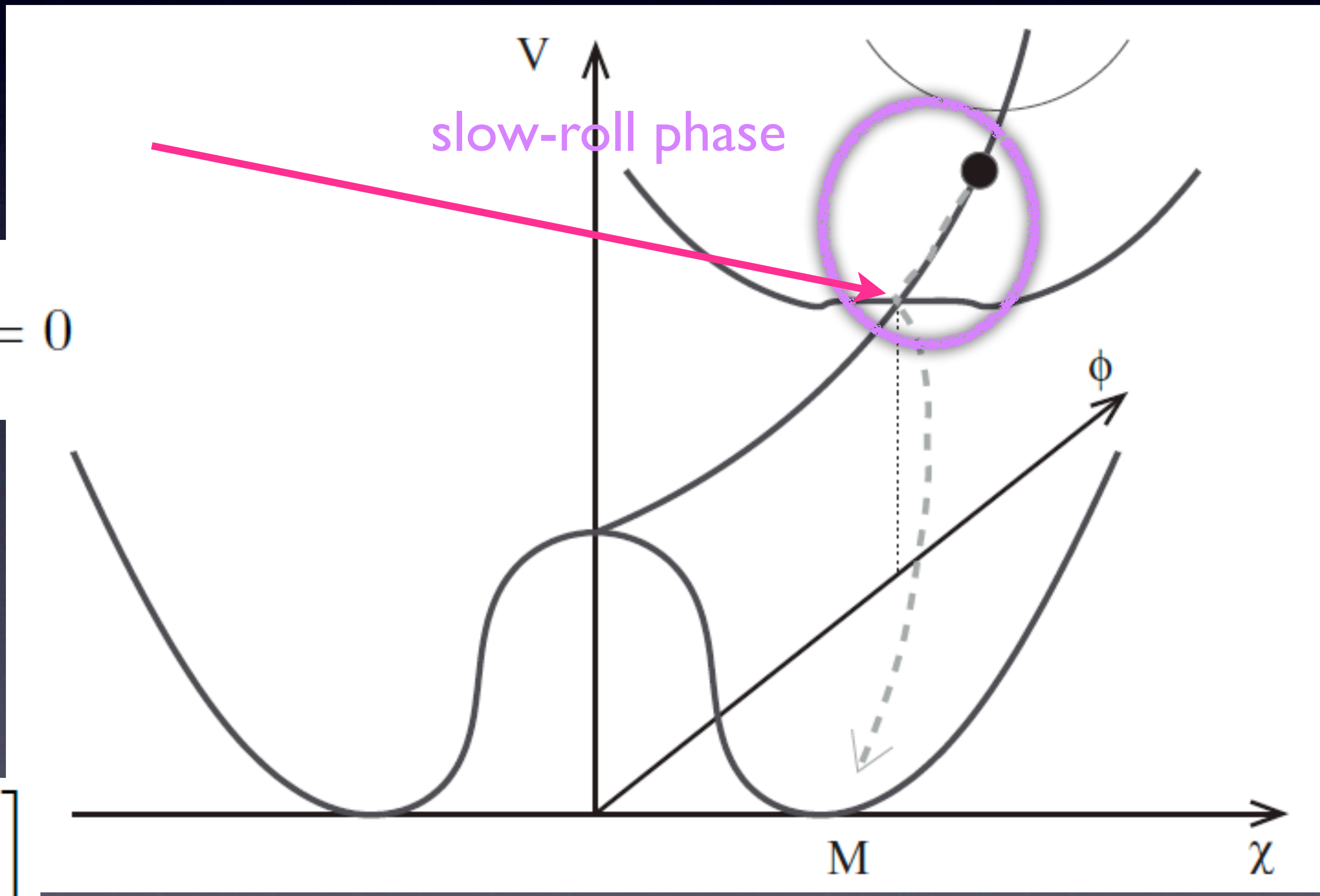
S. Yokoyama & J. Soda [astro-ph: 0805.4265]

Set the potential looks like an Abelian Higgs model in unitary gauge:

$$V(\phi, \chi, A^i) = \frac{\lambda}{4}(\chi^2 - v^2)^2 + \frac{1}{2}g^2\phi^2\chi^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}h^2 A^\mu A_\mu \chi^2$$

At the end of inflation $t = t_e$, effective mass squared of the waterfall field vanishes as

$$m_\chi^2 \equiv \frac{\partial^2 V}{\partial \chi^2} = -\lambda v^2 + g^2\phi_e^2 + h^2 A^i A_i = 0$$



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From this, obtain the relations:

$$\hat{q}^i = -\hat{A}^i\ , \quad \hat{q}^{ij} = -\frac{1}{\phi_e}\left[\left(\frac{g\phi_e}{hA}\right)^2 \delta^{ij} + \hat{A}^i\hat{A}^j\right]$$

$$\beta \simeq \frac{1}{f_e^2}\left(\frac{h^2 A}{g^2\phi_e}\right)^2$$

An Inflation model with a preferred direction

S. Yokoyama & J. Soda [astro-ph: 0805.4265]

Neglecting the terms suppressed by the slow-roll parameters, the primordial bispectrum of curvature perturbations is given by

$$F_\zeta(\boldsymbol{k_1}, \boldsymbol{k_2}, \boldsymbol{k_3}) = C P_\zeta^{\mathrm{iso}}(k_1) P_\zeta^{\mathrm{iso}}(k_2) \hat{A}^a \hat{A}^b \delta^{cd} P_{ac}(\hat{\boldsymbol{k_1}}) P_{bd}(\hat{\boldsymbol{k_2}}) + 2 \text{ perms.}$$

$$C \equiv -g_\beta^2 \frac{\phi_{\mathrm{e}}}{N_{\mathrm{e}}} \left(\frac{g}{hA}\right)^2 .$$

✤Observational bound:

- $g_\beta < O(0.1)$
- $N_e^{-1} \sim \sqrt{\text{slow-roll parameter}} \sim O(0.1)$

Choosing g, h, A, λ, v and reaching $(g/hA)^2 \varphi_e \gg 1$, $C > O(1)$

CMB bispectrum, in which the rotational invariance is violated, may be observed!

CMB Bispectrum from curvature perturbations

$$\left\langle \prod_{n=1}^{3} a_{X_n,\ell_n m_n} \right\rangle = \left[\prod_{n=1}^{3} 4\pi(-i)^{\ell_n} \int_0^\infty \frac{k_n^2 dk_n}{(2\pi)^3} \mathcal{T}_{X_n,\ell_n}(k_n) \right] \left\langle \prod_{n=1}^{3} \zeta_{\ell_n m_n}(k_n) \right\rangle$$

$$\left\langle \prod_{n=1}^{3} \zeta_{\ell_n m_n}(k_n) \right\rangle = \left[\prod_{n=1}^{3} \int d^2\hat{\boldsymbol{k}}_{\boldsymbol{n}} Y^*_{\ell_n m_n}(\hat{\boldsymbol{k}}_{\boldsymbol{n}}) \right] (2\pi)^3 \delta\left(\sum_{n=1}^{3} \boldsymbol{k}_{\boldsymbol{n}} \right) F_\zeta(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$$

~~~~~~~~~~~ *calculation* ~~~~~~~~~~~

1. expand all angular dependencies with ${}_sY_{lm}$

$$\hat{A}^a \hat{A}^b \delta^{cd} P_{ac}(\hat{\boldsymbol{k}}_1) P_{bd}(\hat{\boldsymbol{k}}_2) = -4\left(\frac{4\pi}{3}\right)^3 \sum_{L,L',L_A=0,2} I^{01-1}_{L11} I^{01-1}_{L'11} I^{000}_{11L_A} \left\{ \begin{array}{ccc} L & L' & L_A \\ 1 & 1 & 1 \end{array} \right\} \times \sum_{MM'M_A} Y^*_{LM}(\hat{\boldsymbol{k}}_1) Y^*_{L'M'}(\hat{\boldsymbol{k}}_2) Y^*_{L_A M_A}(\hat{\boldsymbol{A}}) \left(\begin{array}{ccc} L & L' & L_A \\ M & M' & M_A \end{array} \right)$$

$$\delta\left(\sum_{n=1}^{3} \boldsymbol{k}_{\boldsymbol{n}} \right) = 8 \int_0^\infty y^2 dy \left[\prod_{n=1}^{3} \sum_{L_n M_n} (-1)^{L_n/2} j_{L_n}(k_n y) Y^*_{L_n M_n}(\hat{\boldsymbol{k}}_{\boldsymbol{n}}) \right] I^{0\,0\,0}_{L_1 L_2 L_3} \left(\begin{array}{ccc} L_1 & L_2 & L_3 \\ M_1 & M_2 & M_3 \end{array} \right)$$
~~~~~~~~~~~

2. express their integrals with the Wigner symbols

$$\int d^2\hat{\boldsymbol{k}}_1 Y^*_{\ell_1 m_1} Y^*_{L_1 M_1} Y^*_{LM} = I^{0\ 0\ 0}_{\ell_1 L_1 L} \left(\begin{array}{ccc} \ell_1 & L_1 & L \\ m_1 & M_1 & M \end{array} \right)$$

$$\int d^2\hat{\boldsymbol{k}}_2 Y^*_{\ell_2 m_2} Y^*_{L_2 M_2} Y^*_{L' M'} = I^{0\ 0\ 0}_{\ell_2 L_2 L'} \left(\begin{array}{ccc} \ell_2 & L_2 & L' \\ m_2 & M_2 & M' \end{array} \right)$$

$$\int d^2\hat{\boldsymbol{k}}_3 Y^*_{\ell_3 m_3} Y^*_{L_3 M_3} = (-1)^{m_3} \delta_{L_3,\ell_3} \delta_{M_3,-m_3} .$$

$$I^{s_1 s_2 s_3}_{l_1 l_2 l_3} \equiv \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \left(\begin{array}{ccc} l_1 & l_2 & l_3 \\ s_1 & s_2 & s_3 \end{array} \right)$$

Then, the bispectrum of $\zeta_{\ell m}$ is expressed as

$$\begin{aligned} \left\langle \prod_{n=1}^{3} \zeta_{\ell_n m_n}(k_n) \right\rangle &= -(2\pi)^3 8 \int_0^\infty y^2 dy \sum_{L_1 L_2} (-1)^{\frac{L_1+L_2+\ell_3}{2}} I^{0\ 0\ 0}_{L_1 L_2 \ell_3} \\ &\times P^{\rm iso}_\zeta(k_1) j_{L_1}(k_1 y) P^{\rm iso}_\zeta(k_2) j_{L_2}(k_2 y) C j_{\ell_3}(k_3 y) \\ &\times 4 \left(\frac{4\pi}{3}\right)^3 (-1)^{m_3} \sum_{L,L',L_A=0,2} I^{01-1}_{L11} I^{01-1}_{L'11} \\ &\times I^{0\ 0\ 0}_{\ell_1 L_1 L} I^{0\ 0\ 0}_{\ell_2 L_2 L'} I^{000}_{11 L_A} \left\{ \begin{array}{ccc} L & L' & L_A \\ 1 & 1 & 1 \end{array} \right\} \\ &\times \sum_{M_1 M_2 M M' M_A} Y^*_{L_A M_A}(\hat{\boldsymbol{A}}) \left(\begin{array}{ccc} L_1 & L_2 & \ell_3 \\ M_1 & M_2 & -m_3 \end{array} \right) \\ &\times \left(\begin{array}{ccc} \ell_1 & L_1 & L \\ m_1 & M_1 & M \end{array} \right) \left(\begin{array}{ccc} \ell_2 & L_2 & L' \\ m_2 & M_2 & M' \end{array} \right) \left(\begin{array}{ccc} L & L' & L_A \\ M & M' & M_A \end{array} \right) + 2 \text{ perms.} \end{aligned}$$

$$\neq (2\pi)^3 \mathcal{F}_{\ell_1 \ell_2 \ell_3}(k_1, k_2, k_3) \left(\begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{array} \right)$$

evidence of violation of the rotational invariance!!

3. Setting the coordinate as $\hat{A} = \hat{z}$

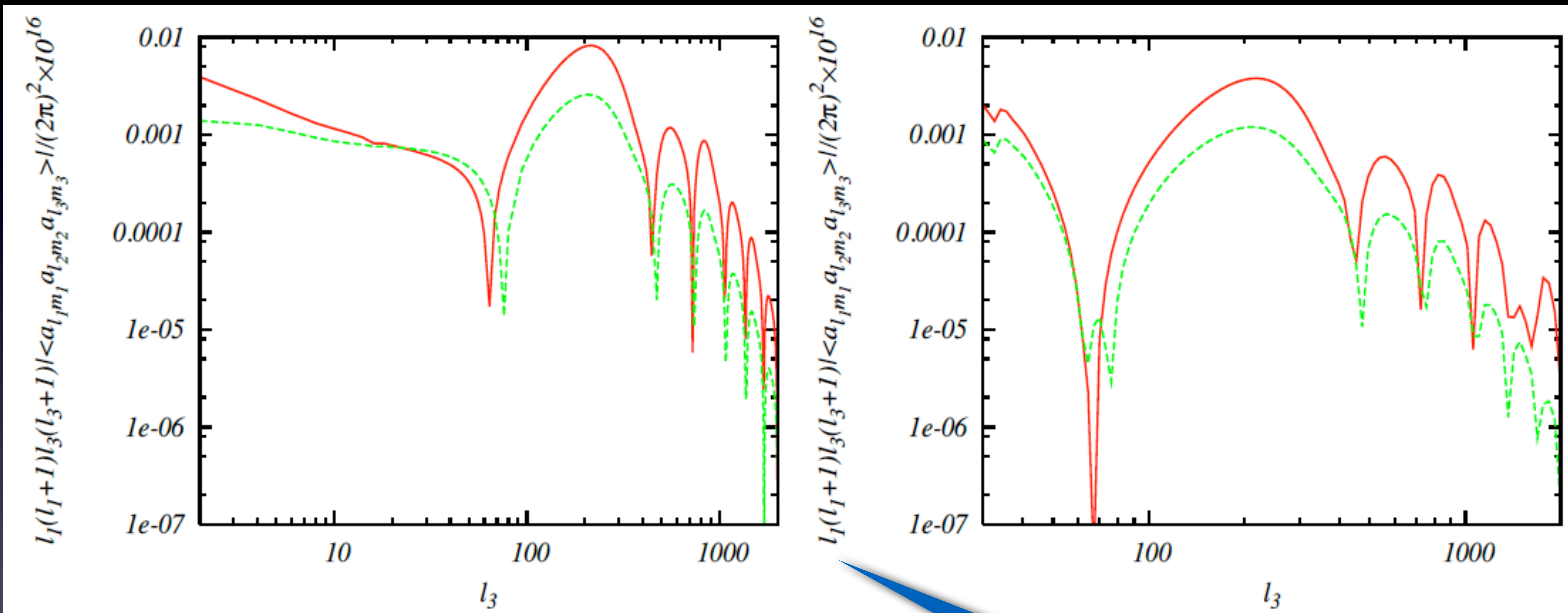
and using the relation: $Y^*_{L_A M_A}(\hat{z}) = \sqrt{(2L_A+1)/(4\pi)}\delta_{M_A,0}$

the CMB bispectrum is explicitly written as

$$\left\langle \prod_{n=1}^{3} a_{X_n,\ell_n m_n} \right\rangle = -\int_0^\infty y^2 dy \left[\prod_{n=1}^{3} \frac{2}{\pi} \int_0^\infty k_n^2 dk_n \mathcal{T}_{X_n,\ell_n}(k_n) \right]$$
$$\times \sum_{L_1 L_2} (-1)^{\frac{\ell_1+\ell_2+L_1+L_2}{2}+\ell_3} I^{0\ 0\ 0}_{L_1 L_2 \ell_3}$$
$$\times P_\zeta^{\rm iso}(k_1) j_{L_1}(k_1 y) P_\zeta^{\rm iso}(k_2) j_{L_2}(k_2 y) C j_{\ell_3}(k_3 y)$$
$$\times 4 \left(\frac{4\pi}{3}\right)^3 (-1)^{m_3} \sum_{L,L',L_A=0,2} I^{01-1}_{L11} I^{01-1}_{L'11}$$
$$\times I^{0\ 0\ 0}_{\ell_1 L_1 L} I^{0\ 0\ 0}_{\ell_2 L_2 L'} I^{000}_{11L_A} \left\{ \begin{matrix} L & L' & L_A \\ 1 & 1 & 1 \end{matrix} \right\}$$
$$\times \sqrt{\frac{2L_A+1}{4\pi}} \sum_{M=-2}^{2} \left(\begin{matrix} L_1 & L_2 & \ell_3 \\ -m_1 - M & -m_2 + M & -m_3 \end{matrix} \right)$$
$$\times \left(\begin{matrix} \ell_1 & L_1 & L \\ m_1 & -m_1 - M & M \end{matrix} \right) \left(\begin{matrix} \ell_2 & L_2 & L' \\ m_2 & -m_2 + M & -M \end{matrix} \right)$$
$$\times \left(\begin{matrix} L & L' & L_A \\ M & -M & 0 \end{matrix} \right) + 2 \text{ perms.} \,. \quad (3$$

CMB bispectrum for $\ell_1 = \ell_2 = \ell_3$

❖ $(m_1, m_2, m_3) = (0, 0, 0)$ ❖ $(m_1, m_2, m_3) = (10, 20, -30)$



-statistically-anisotropic bispectrum: C = 1

-statistically-isotropic bispectrum: $f_{NL}^{local} = 5$

Overall behavior seems to be in agreement with the isotropic case

Special configuration

If the CMB bispectrum satisfy the rotational invariance as

$$\left\langle \prod_{n=1}^{3} a_{\ell_n m_n} \right\rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1 \ell_2 \ell_3}$$

Due to the triangle inequality, CMB statistically-isotropic bispectrum is exactly zero at multipoles other than

$$|\ell_2 - \ell_3| \le \ell_1 \le \ell_2 + \ell_3$$

On the other hand, Due to the triangle inequality, this bispectrum has nonzero value only for the condition:

$$\sum_{n=1}^{3} \ell_n = \text{even} \ , \quad \sum_{n=1}^{3} m_n = 0 \ ,$$
$$L_1 = |\ell_1 - 2|, \ell_1, \ell_1 + 2 \ , \quad L_2 = |\ell_2 - 2|, \ell_2, \ell_2 + 2$$
$$|L_2 - \ell_3| \le L_1 \le L_2 + \ell_3 \ .$$

+ 2 perm. of ℓ_1, ℓ_2, ℓ_3

Like the off-diagonal component of the CMB power spectrum, CMB statistically-anisotropic bispectrum does not vanish also in the configuration as

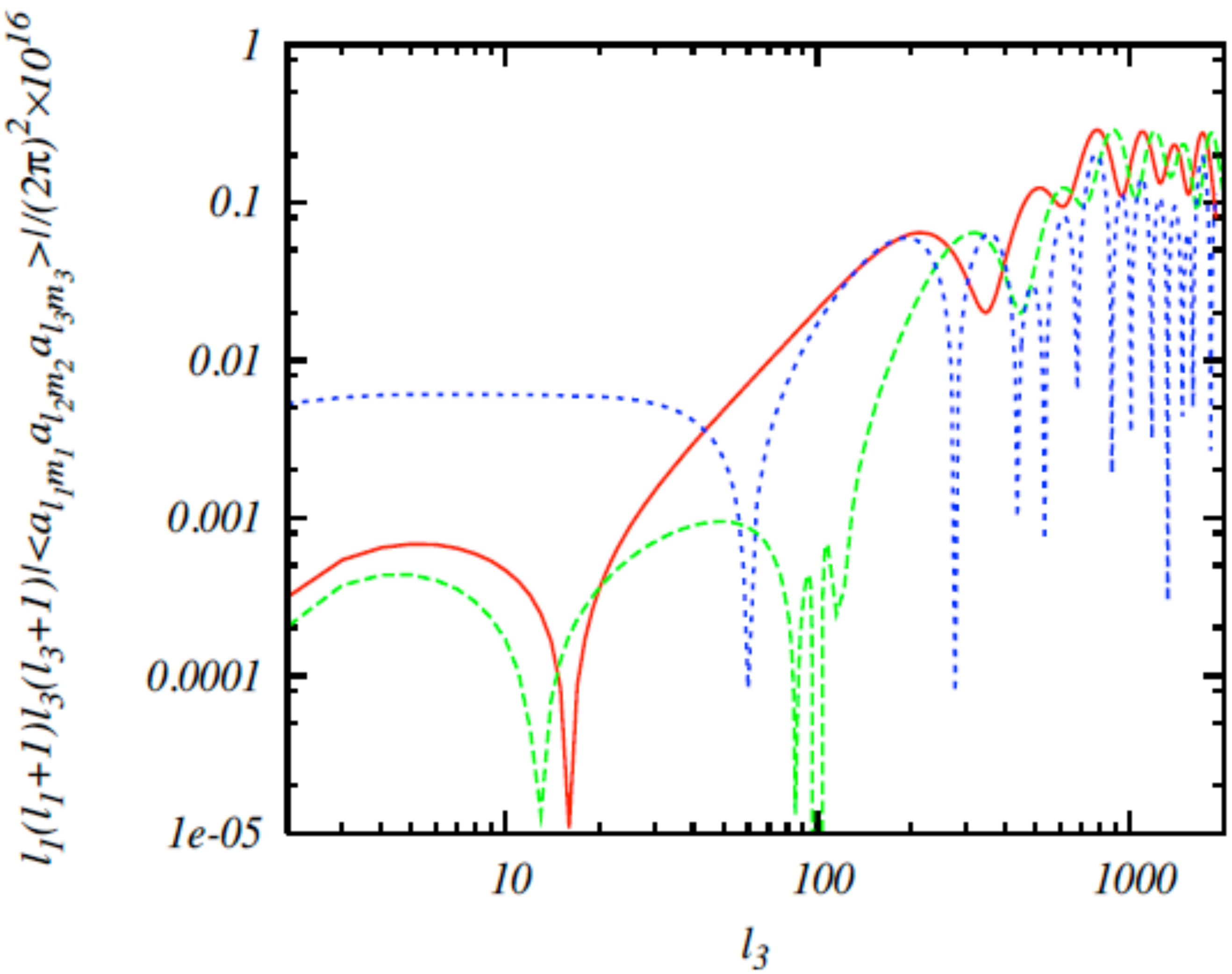
$$\ell_1 = \ell_2 + \ell_3 + 2, |\ell_2 - \ell_3| - 2$$

+ 2 perm. of ℓ_1, ℓ_2, ℓ_3

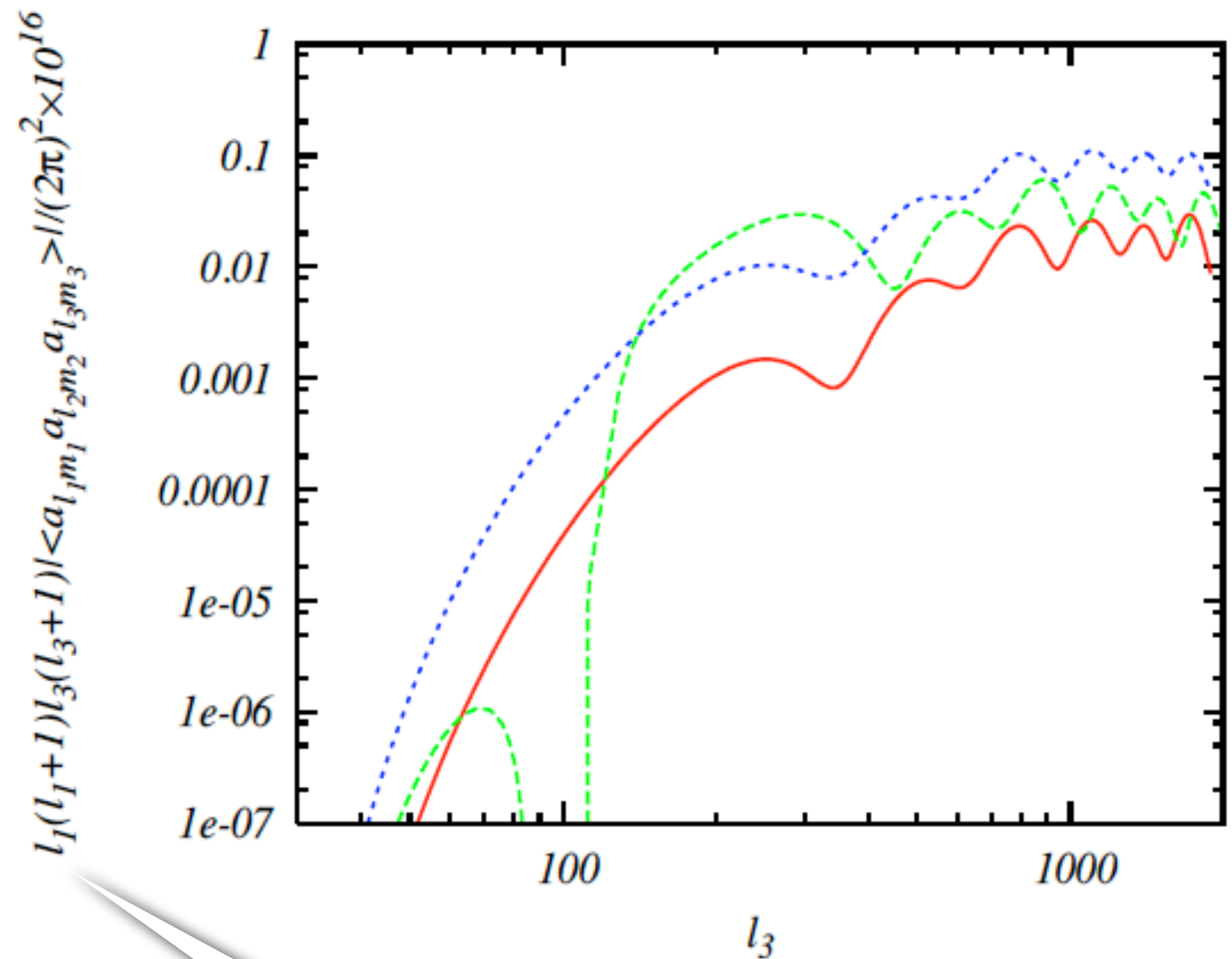
Pure information of the statistical anisotropy!!

CMB bispectrum for different ℓ

- $(m_1, m_2, m_3) = (0, 0, 0)$



- $(m_1, m_2, m_3) = (10, 20, -30)$



Signals of the special configuration are comparable in magnitude to other configuration satisfying the triangle condition such as $\ell_1 = \ell_2 + \ell_3$

- $(\ell_1, \ell_2) = (102 + \ell_3, 100)$
- $(\ell_1, \ell_2) = (|100 - \ell_3| - 2, 100)$
- $(\ell_1, \ell_2) = (100 + \ell_3, 100)$

Summary

- Based on an inflation model which produces the large direction-depending non-Gaussianity of curvature perturbations, we formulate the CMB bispectrum and analyze its behaviors
- There exists the non-vanishing configuration of multipoles which deviates from the triangle condition and these signals are comparable in magnitude to those of the other configuration

if ΔT is invariant under the rotational transformation,

$$\Delta T(R\hat{\mathbf{n}}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(R\hat{\mathbf{n}}) = \sum_{\ell m} a_{\ell m} \sum_{m'} D^{(\ell)}_{m'm}(R^{-1}) Y_{\ell m'}(\hat{\mathbf{n}})$$
$$= \Delta T(\hat{\mathbf{n}}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

$$\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle$$
$$= \sum_{\text{all } m'} \langle a_{l_1 m_1'} a_{l_2 m_2'} a_{l_3 m_3'} \rangle D^{(l_1)}_{m_1' m_1} D^{(l_2)}_{m_2' m_2} D^{(l_3)}_{m_3' m_3}$$
$$= \langle B_{l_1 l_2 l_3} \rangle \sum_{\text{all } m'} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1' & m_2' & m_3' \end{pmatrix}$$
$$\times \sum_{L M M'} \begin{pmatrix} l_1 & l_2 & L \\ m_1' & m_2' & M' \end{pmatrix} \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & M \end{pmatrix} (2L+1) D^{(L)*}_{M'M} D^{(l_3)}_{m_3' m_3}$$
$$= \langle B_{l_1 l_2 l_3} \rangle \sum_{m_3'} \sum_{L M M'} \delta_{l_3 L} \delta_{m_3' M'} \begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & M \end{pmatrix} D^{(L)*}_{M'M} D^{(l_3)}_{m_3' m_3}$$
$$= \langle B_{l_1 l_2 l_3} \rangle \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

If the rotational invariance violates, three azimuthal quantum numbers is not confined in this Wigner-3j symbol!!

Power spectrum of vector fields

✤quantization

$$A_i(\tau, x^i) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{\lambda=1}^{2} \epsilon_{i\lambda}(\mathbf{k}) \left[v_k(\tau)\hat{a}_{\lambda\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + v_k^*(\tau)\hat{a}^\dagger_{\lambda\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \right]$$

$$\sum_{\lambda=1}^{2} \epsilon^i_\lambda(\mathbf{k})\epsilon_{j\lambda}(\mathbf{k}) = \delta^i{}_j - \delta_{j\ell}\frac{k^i k^\ell}{k^2}$$

✤field eq.

$$\psi_k'' + \left(k^2 - \frac{\alpha(\alpha+1)}{\tau^2}\right)\psi_k = 0$$

$\psi_k = a v_k$;

$f \propto a^\alpha$

✤power spectrum:

$$\langle \delta A_i(\mathbf{k}) \delta A_j(\mathbf{k}') \rangle = \frac{|\psi_k|^2}{a^2 f^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \delta^{(3)}(\mathbf{k} + \mathbf{k}')$$

solution:

$$\psi_k = \left(\frac{\pi}{4k}\right)^{\frac{1}{2}} \exp\left[i(\alpha+1)\frac{\pi}{2}\right] (-k\tau)^{1/2} H^{(1)}_{\alpha+1/2}(-k\tau)$$

superhorizon limit:

$$\to \frac{(-ik\tau)^{-1}}{\sqrt{2k}} \left[1 + O\left((-k\tau)^2\right)\right]$$

(α = 1, -2) → $(\delta A)^2 \propto k^{-3}$

$$\to \frac{1}{\sqrt{2k}} e^{-ik\tau}$$

(α = 0, -1) → $(\delta A)^2 \propto k^{-1}$