Construction of gauge-invariant variables for linear-order metric perturbations on general background spacetime

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I. Introduction

- **The higher order perturbation theory in general relativity has very wide physical motivation.**

  - **Cosmological perturbation theory**
    - **Expansion law of inhomogeneous universe (ΛCDM v.s. inhomogeneous cosmology)**
    - **Non-Gaussianity in CMB (beyond WMAP)**

  - **Black hole perturbations**
    - **Radiation reaction effects due to the gravitational wave emission.**
    - **Close limit approximation of black hole - black hole collision (Gleiser, et al. (1996))**

  - **Perturbation of a star (Neutron star)**
    - **Rotation – pulsation coupling (Kojima 1997)**

There are many physical situations to which higher order perturbation theory should be applied.

However, general relativistic perturbation theory requires very delicate treatments of "gauges".

It is worthwhile to formulate the higher-order gauge-invariant perturbation theory from general point of view.

- According to this motivation, we have been formulating the general relativistic second-order perturbation theory in a gauge-invariant manner.
  - **General framework**:
    - Framework of higher-order gauge-invariant perturbations:
      - K.N. PTP**110** (2003), 723; *ibid.* **113** (2005), 413.
    - Construction of gauge-invariant variables for the linear order metric perturbation:
      - K.N. CQG**28** (2011), 122001; 1105.4007[gr-qc].
  - **Application to cosmological perturbation theory** :
    - Einstein equations :
      - K.N. PRD**74** (2006), 101301R; PTP**117** (2007), 17.
    - Equations of motion for matter fields:
      - K.N. PRD**80** (2009), 124021.
    - Consistency of the 2nd order Einstein equations :
      - K.N. PTP**121** (2009), 1321.
    - Summary of current status of this formulation:
      - K.N. Adv. in Astron. **2010** (2010), 576273.
    - Comparison with a different formulation:
      - A.J. Christopherson, K. Malik, D.R. Matravers, and K.N. arXiv:1101.3525 [astro-ph.CO]

Our general framework of the second-order gauge invariant perturbation theory is based on a single assumption.

- metric perturbation : metric on PS : $\bar{g}_{ab}$, metric on BGS : g_{ab}

  metric expansion : $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3)$

- **linear order (assumption, decomposition hypothesis) :**

Suppose that the linear order perturbation h_{ab} is decomposed as

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

so that the variable $\mathcal{H}_{ab}$ and X^a are the gauge invariant and the gauge variant parts of h_{ab} , respectively.

These variables are transformed as

$$ {}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0 \qquad {}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi_1^a $$

under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$.

- In cosmological perturbations, this is **almost** correct and we may choose $\mathcal{H}_{ab}$ as (longitudinal gauge, J. Bardeen (1980))

$$\mathcal{H}_{\eta\eta} = -2a^2 \overset{(1)}{\Phi}, \quad \mathcal{H}_{i\eta} = a^2 \overset{(1)}{\nu_i}, \quad \mathcal{H}_{ij} = -2a^2 \overset{(1)}{\Psi} + a^2 \overset{(1)}{\chi_{ij}},$$

Problems in the decomposition hypothesis

- Even in cosmological perturbations, ...
  - Background metric : $g_{ab} = a^2(\eta)\left(-(d\eta)_a(d\eta)_b + \gamma_{ij}(dx^i)_a(dx^j)_b\right)$

    γ_{ij} : metric on maximally symmetric 3-space
  - **Zero-mode problem** :

$$h_{ab} =: h_{\eta\eta}(d\eta)_a(d\eta)_b + 2h_{\eta i}(d\eta)_{(a}(dx^i)_{b)} + h_{ij}(dx^i)_a(dx^j)_b$$

$$\begin{aligned} h_{\eta i} &= D_i h_{(VL)} + h_{(V)i}, \quad D^i h_{(V)i} = 0, \\ h_{ij} &= a^2 h_L \gamma_{ij} + a^2 h_{(T)ij}, \quad h_{(T)}{}^i{}_i := \gamma^{ij} h_{(T)ij} = 0, \\ h_{(T)ij} &= \left(D_i D_j - \frac{1}{3}\gamma_{ij}\Delta\right) h_{TL} + 2D_{(i}h_{(TV)j)} + h_{(TT)ij}, \\ D^i h_{(TV)i} &= 0, \quad D^i h_{(TT)ij} = 0, \quad D_i\gamma_{ij} := 0, \quad \Delta := D^i D_i. \end{aligned}$$

  - This decomposition is based on the existence of Green functions

    $\Delta^{-1}, \quad (\Delta + 2K)^{-1}, \quad (\Delta + 3K)^{-1}.$ K: curvature constant associated with γ_{ij}
  - In our formulation, we ignored the modes (zero modes) which belong to the kernel of the operators Δ , $(\Delta + 2K)$, $(\Delta + 3K)$.
  - **How to include these zero modes into our consideration?**
- On general background spacetime, --> **Generality problem** :
  - **Is the decomposition hypothesis $h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$ also correct in general background spacetime?**

- In this talk, ...
  - We show an outline of a proof of this generality problem using ADM decomposition.

V. Summary

- In this talk, ...
  - We show an outline of the proof of this generality problem using ADM decomposition.
    - In our proof, we assumed the existence of Green functions of two derivative operators

$$\Delta := D^i D_i, \quad \mathcal{F} := \Delta - \frac{2}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right) D^i - 2D^i \left\{\frac{1}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right)\right\},$$

and the existence and the uniqueness of the solution to the equation

$$\mathcal{D}_j{}^k A_k + D^m \left[\frac{2}{\alpha}\tilde{K}_{mj}\left\{\mathcal{F}^{-1} D^k \left(\frac{2}{\alpha} M_k{}^l A_l - \partial_t A_k - \beta^k A_k\right)\right\}\right] = L_j, \quad \tilde{K}_{ij} := K_{ij} - \frac{1}{n} q_{ij} K.$$

$$\Delta := D^i D_i, \quad \mathcal{D}_j{}^l := q_j^l \Delta + \left(1 - \frac{2}{n}\right) D_j D^l + R_j{}^l$$

$R_j{}^l$: Ricci curvature on Σ .

---> **The zero-mode problem is more delicate in general case!!**

We may say that the decomposition hypothesis

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

is almost correct for the linear metric perturbation on general background spacetime.

II. "Gauge" in general relativity

(R.K. Sachs (1964).)

- There are two kinds of "gauge" in general relativity.
  - The concepts of these two "gauge" are closely related to the general covariance.
  - "General covariance" :

    There is no preferred coordinate system in nature.

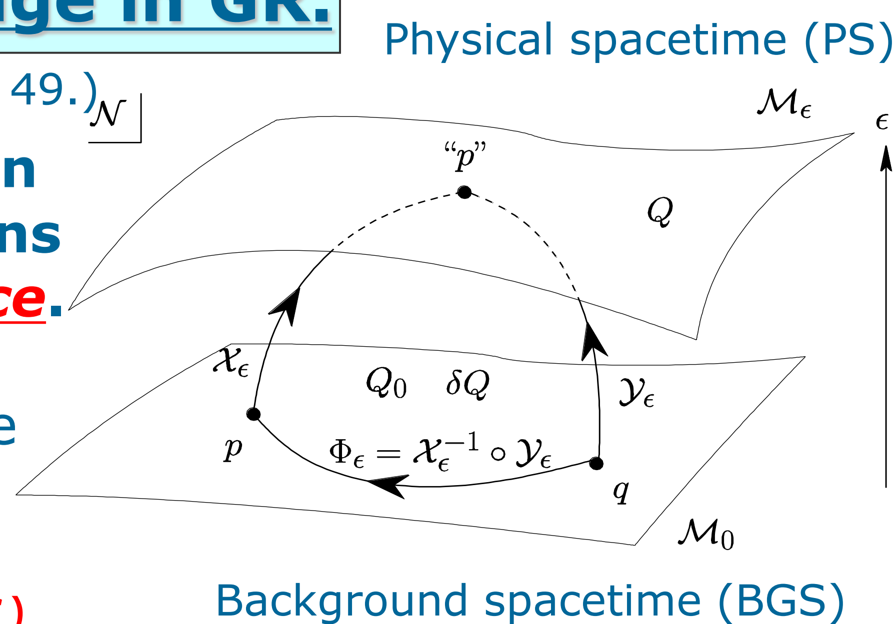
- **The first kind "gauge"** is a coordinate system on a single spacetime manifold.

- **The second kind "gauge"** appears in the perturbation theory.

  This is a point identification between the physical spacetime and the background spacetime.
  - To explain this second kind "gauge", we have to remind what we are doing in perturbation theory.

III. The second kind gauge in GR.

(Stewart and Walker, PRSL **A341** (1974), 49.)



- "***Gauge degree of freedom***" **in general relativistic perturbations arises due to *general covariance*.**

- In any perturbation theories, we always treat two spacetimes :
  - *Physical Spacetime (PS);*
  - *Background Spacetime (BGS).*

- In perturbation theories, we always write equations like

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p)$$

Through this equation, we always identify the points on these two spacetimes and this identification is called "gauge choice" in perturbation theory.

Gauge transformation rules of each order

- **Expansion of gauge choices :**

We assume that each gauge choice is an exponential map.

$$Q_{\mathcal{X}} = \mathcal{X}^{*}_{\epsilon} Q = Q + \epsilon \pounds_{u} Q + \frac{1}{2}\epsilon^{2} \pounds^{2}_{u} Q + O(\epsilon^{3})$$

$$Q_{\mathcal{Y}} = \mathcal{Y}^{*}_{\epsilon} Q = Q + \epsilon \pounds_{v} Q + \frac{1}{2}\epsilon^{2} \pounds^{2}_{v} Q + O(\epsilon^{3})$$

$$\Phi^{*}_{\epsilon} Q = \left(\mathcal{X}^{-1}_{\epsilon} \circ \mathcal{Y}_{\epsilon}\right)^{*} Q = Q + \epsilon \pounds_{\xi_1} Q + \frac{1}{2}\epsilon^{2} \left(\pounds_{\xi_2} + \pounds^{2}_{\xi_1}\right) Q + O(\epsilon^{3})$$

(Sonego and Bruni, CMP, **193** (1998), 209.)

-------> $\xi_1 = u - v, \quad \xi_2 = [u, v]$

- **Expansion of the variable :** $Q = Q_0 + \epsilon Q_1 + \frac{1}{2} Q_2 + O(\epsilon^3)$
- **Order by order gauge transformation rules :**

$${}_{\mathcal{Y}}Q_1 - {}_{\mathcal{X}}Q_1 = \pounds_{\xi_1} Q_0$$

$${}_{\mathcal{Y}}Q_2 - {}_{\mathcal{X}}Q_2 = 2\pounds_{\xi_1} {}_{\mathcal{X}}Q_1 + \left(\pounds_{\xi_2} + \pounds^{2}_{\xi_1}\right) Q_0$$

Through these understanding of gauges
and the gauge-transformation rules,
we developed second-order gauge-invariant perturbation theory.

III. Construction of gauge invariant variables in higher order perturbations

- metric perturbation : metric on PS : $\bar{g}_{ab}$, metric on BGS : g_{ab}

  metric expansion : $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3)$

  Our general framework of the second-order gauge invariant perturbation theory **WAS** based on a single assumption.

○ **linear order (decomposition hypothesis) :**

Suppose that the linear order perturbation h_{ab} is decomposed as

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

so that the variable $\mathcal{H}_{ab}$ and X^a are the gauge invariant and the gauge variant parts of h_{ab} , respectively.

These variables are transformed as

$${}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0 \qquad {}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi_1^a$$

under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$.

Second order :

Once we accept the above assumption for the linear order metric perturbation h_{ab} , we can always decompose the second order metric perturbations l_{ab} as follows :

$$l_{ab} =: \mathcal{L}_{ab} + 2\pounds_X h_{ab} + \left(\pounds_Z - \pounds_X^2\right) g_{ab}$$

where $\mathcal{L}_{ab}$ is gauge invariant part and Z_a is gauge variant part.

Under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$, the vector field Z_a is transformed as $\;{}_{\mathcal{Y}}Z^a - {}_{\mathcal{X}}Z^a = \xi_2^a + [\xi_1, X]^a$

○ Perturbations of an arbitrary matter field Q :

Using gauge variant part of the metric perturbation of each order, gauge invariant variables for an arbitrary tensor fields Q other than metric are defined by

- First order perturbation of Q :

$$\mathcal{Q}_1 := Q_1 - \pounds_X Q_0$$

- Second order perturbation of Q :

$$\mathcal{Q}_2 := Q_2 - 2\pounds_X Q_1 - \left\{\pounds_Z - \pounds_X^2\right\} Q_0$$

These implies that each order perturbation of an arbitrary field is always decomposed as

$$Q_1 = \boxed{\mathcal{Q}_1} + \boxed{\pounds_X Q_0}$$

$$Q_2 = \boxed{\mathcal{Q}_2} + \boxed{2\pounds_X Q_1 + \left\{\pounds_Z - \pounds_X^2\right\} Q_0}$$

(green box) : gauge invariant part

(yellow box) : gauge variant part

III. Construction of gauge invariant variables in higher-order perturbations

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  Our general framework of the second-order gauge invariant perturbation theory **WAS** based on a single assumption.

- **linear order (decomposition hypothesis) :**

Suppose that the linear order perturbation h_{ab} is decomposed as

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

so that the variable $\mathcal{H}_{ab}$ and X^a are the gauge invariant and the gauge variant parts of h_{ab} , respectively.

These variables are transformed as

$$_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0 \qquad _{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi^a_1$$

under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$.

This decomposition hypothesis is an important premise of our general framework of higher-order gauge-invariant perturbation theory.

IV. Construction of gauge-invariant variables for linear metric perturbations on general background spacetime

(K.N. CQG **28** (2011), 122001; 1105.4007[gr-qc].)

- metric perturbation : metric on PS : $\bar{g}_{ab}$, metric on BGS : g_{ab}

  metric expansion : $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3)$.

- ADM decomposition of BGS : $\mathcal{M}_0 = \mathbb{R} \times \Sigma$, $\dim(\Sigma) = n$

$$g_{ab} = -\alpha^2 (dt)_a (dt)_b + q_{ij} (dx^i + \beta^i dt)_a (dx^j + \beta^j dt)_b, \quad \mathrm{sign}(q_{ij}) = (+, \cdots, +).$$

- Gauge-transformation for the linear metric perturbation h_{ab}

$${}_{\mathcal{Y}}h_{ab} - {}_{\mathcal{X}}h_{ab} = \pounds_{\xi} g_{ab}$$

$$\xi_a = \xi_t (dt)_a + \xi_i (dx^i)_a$$

$$h_{ab} = h_{tt} (dt)_a (dt)_b + 2 h_{ti} (dt)_{(a} (dx^i)_{b)} + h_{ij} (dx^i)_a (dx^j)_b.$$

$$\begin{aligned}
{}_{\mathcal{Y}}h_{tt} - {}_{\mathcal{X}}h_{tt} &= \boxed{2\partial_t \xi_t} - \frac{2}{\alpha}\left(\partial_t \alpha + \beta^i D_i \alpha - \beta^j \beta^i K_{ij}\right) \xi_t \\
&\quad - \frac{2}{\alpha}\left(\beta^i \beta^k \beta^j K_{kj} - \beta^i \partial_t \alpha + \alpha q^{ij} \partial_t \beta_j \right. \\
&\quad \left. + \alpha^2 D^i \alpha - \alpha \beta^k D^i \beta_k - \beta^i \beta^j D_j \alpha \right) \xi_i, \\
{}_{\mathcal{Y}}h_{ti} - {}_{\mathcal{X}}h_{ti} &= \boxed{\partial_t \xi_i + D_i \xi_t} - \frac{2}{\alpha}\left(D_i \alpha - \beta^j K_{ij} \right) \xi_t - \frac{2}{\alpha} M_i{}^j \xi_j, \\
{}_{\mathcal{Y}}h_{ij} - {}_{\mathcal{X}}h_{ij} &= \boxed{2 D_{(i} \xi_{j)}} + \frac{2}{\alpha} K_{ij} \xi_t - \frac{2}{\alpha} \beta^k K_{ij} \xi_k, \\
M_i{}^j &:= -\alpha^2 K^j{}_i + \beta^j \beta^k K_{ki} - \beta^j D_i \alpha + \alpha D_i \beta^j.
\end{aligned}$$

extrinsic curvature:

$K^j{}_i := q^{jk} K_{ki}$

covariant derivative:

$D_i q_{jk} = 0$

Through the similar logic to the simple case $\alpha = 1, \quad \beta^i = 0$ in the paper [K.N. CQG28 (2011),122001.], we can derive the following decomposition :

$$\begin{aligned}
h_{ti} &= D_i h_{(VL)} + h_{(V)i} - \frac{2}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right)\left\{h_{(VL)} - \Delta^{-1} D^k \partial_t h_{(TV)k}\right\} - \frac{2}{\alpha} M_i{}^k h_{(TV)k}, \\
h_{ij} &= \frac{1}{n} q_{ij} h_{(L)} + h_{(T)ij} + \frac{2}{\alpha} K_{ij}\left\{h_{(VL)} - \Delta^{-1} D^k \partial_t h_{(TV)k}\right\} - \frac{2}{\alpha}\beta^k K_{ij} h_{(TV)k}, \\
h_{(T)ij} &= D_i h_{(TV)j} + D_j h_{(TV)i} - \frac{2}{n} q_{ij} D^l h_{(TV)l} + h_{(TT)ij}, \\
D^i h_{(V)i} &= 0, \quad q^{ij} h_{(TT)ij} = 0, \quad D^i h_{(TT)ij} = 0.
\end{aligned}$$

The inverse relations of these decompositions are guaranteed by the existence of Green functions of elliptic derivative operators

$$\Delta := D^i D_i, \quad \mathcal{F} := \Delta - \frac{2}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right) D^i - 2D^i\left\{\frac{1}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right)\right\},$$

and the existence and the uniqueness of the integro-differential equation for a vector field A_k,

$$\mathcal{D}_j{}^k A_k + D^m\left[\frac{2}{\alpha}\tilde{K}_{mj}\left\{\mathcal{F}^{-1} D^k\left(\frac{2}{\alpha} M_k{}^l A_l - \partial_t A_k - \beta^k A_k\right)\right\}\right] = L_j, \qquad \tilde{K}_{ij} := K_{ij} - \frac{1}{n} q_{ij} K.$$

$\Delta := D^i D_i, \quad \mathcal{D}_j{}^l := q_j^l \Delta + \left(1 - \frac{2}{n}\right) D_j D^l + R_j{}^l$ $\qquad R_j{}^l$: Ricci curvature on Σ.

- We can derive gauge-transformation rules for variables as

$$
\begin{aligned}
{}_{\mathcal{Y}}h_{tt} - {}_{\mathcal{X}}h_{tt} &= 2\partial_t\xi_t - \frac{2}{\alpha}\left(\partial_t\alpha + \beta^i D_i\alpha - \beta^j\beta^i K_{ij}\right)\xi_t \\
&\quad -\frac{2}{\alpha}\Big(\beta^i\beta^k\beta^j K_{kj} - \beta^i\partial_t\alpha + \alpha q^{ij}\partial_t\beta_j + \alpha^2 D^i\alpha \\
&\qquad -\alpha\beta^k D^i\beta_k - \beta^i\beta^j D_j\alpha\Big)\xi_i, \\
{}_{\mathcal{Y}}h_{(VL)} - {}_{\mathcal{X}}h_{(VL)} &= \xi_t + \Delta^{-1}D^k\partial_t\xi_k, \\
{}_{\mathcal{Y}}h_{(V)i} - {}_{\mathcal{X}}h_{(V)i} &= \partial_t\xi_i - D_i\Delta^{-1}D^k\partial_t\xi_k, \\
{}_{\mathcal{Y}}h_{(L)} - {}_{\mathcal{X}}h_{(L)} &= 2D^i\xi_i, \\
{}_{\mathcal{Y}}h_{(TV)i} - {}_{\mathcal{X}}h_{(TV)i} &= \xi_i, \\
{}_{\mathcal{Y}}h_{(TT)ij} - {}_{\mathcal{X}}h_{(TT)ij} &= 0.
\end{aligned}
$$

- Gauge-variant variables : ${}_{\mathcal{Y}}X_t - {}_{\mathcal{X}}X_t = \xi_t, \quad {}_{\mathcal{Y}}X_i - {}_{\mathcal{X}}X_i = \xi_i.$

$$X_t := h_{(VL)} - \Delta^{-1}D^k\partial_t h_{(TV)k}, \quad X_i := h_{(TV)i}.$$

Gauge-invariant variables :

$$
\begin{aligned}
-2\Phi &:= h_{tt} - 2\partial_t X_t + \frac{2}{\alpha}\left(\partial_t\alpha + \beta^i D_i\alpha - \beta^j\beta^i K_{ij}\right) X_t \\
&\quad + \frac{2}{\alpha}\left(\beta^i\beta^k\beta^j K_{kj} - \beta^i\partial_t\alpha + \alpha q^{ij}\partial_t\beta_j + \alpha^2 D^i\alpha - \alpha\beta^k D^i\beta_k - \beta^i\beta^j D_j\alpha\right) X_i, \\
-2n\Psi &:= h_{(L)} - 2D^i X_i = h_{(L)} - 2D^i h_{(TV)i}, \\
\nu_i &:= h_{(V)i} - \partial_t h_{(TV)i} + D_i\Delta^{-1}D^k\partial_t h_{(TV)k}, \\
\chi_{ij} &:= h_{(TT)ij}.
\end{aligned}
$$

$$
D^i\nu_i = 0, \quad q^{ij}\chi_{ij} = 0, \quad D^i\chi_{ij} = 0, \quad \chi_{ij} = \chi_{ji}.
$$

Definitions of gauge-invariant and gauge-variant parts :

$$
\mathcal{H}_{ab} := -2\Phi(d\eta)_a(d\eta)_b + 2\nu_i(d\eta)_{(a}(dx^i)_{b)} + \left(-2\Psi q_{ij} + \chi_{ij}\right)(dx^i)_{(a}(dx^j)_{b)},
$$

$$
X_a := X_t(d\eta)_a + X_i(dx^i)_a.
$$

In terms of these variables, the original components h_{tt}, h_{ti}, and h_{ij} of the linear metric perturbation h_{ab} are summarized in the covariant form :

$$
h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}
$$

V. Summary

- In this talk, ...
  - We show an outline of the proof of this generality problem using ADM decomposition.
    - In our proof, we assumed the existence of Green functions of two derivative operators

$$\Delta := D^i D_i, \quad \mathcal{F} := \Delta - \frac{2}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right) D^i - 2D^i \left\{ \frac{1}{\alpha}\left(D_i\alpha - \beta^j K_{ij}\right) \right\},$$

and the existence and the uniqueness of the solution to the equation

$$\mathcal{D}_j{}^k A_k + D^m \left[\frac{2}{\alpha}\tilde{K}_{mj} \left\{ \mathcal{F}^{-1} D^k \left(\frac{2}{\alpha} M_k{}^l A_l - \partial_t A_k - \beta^k A_k \right) \right\} \right] = L_j, \quad \tilde{K}_{ij} := K_{ij} - \frac{1}{n} q_{ij} K.$$

$$\Delta := D^i D_i, \quad \mathcal{D}_j{}^l := q_j^l \Delta + \left(1 - \frac{2}{n}\right) D_j D^l + R_j{}^l$$

$R_j{}^l$: Ricci curvature on Σ.

---> **The zero-mode problem is more delicate in general case!!**

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is almost correct for the linear metric perturbation on general background spacetime.