Deformation of Codimension-2 Surfaces and Horizon Thermodynamics

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Contents

Submanifold

For a spacelike submanifold, from the submanifold theory, one can always decompose the metric of the spacetime into

$$g_{ab} = h_{ab} + q_{ab} \,, \tag{1}$$

The *second fundamental tensor* ${K_{ab}}^c$ is defined as

$${K_{ab}}^c = {q_a}^d {q_b}^e \nabla_d {q_e}^c \,. \tag{2}$$

It can be defined without introducing any local frame of the spacetime (B.Carter, 1992).

The second fundamental tensor can be decomposed into a traceless part (${C_{ab}}^c$) and a trace part (K^c), i.e.,

$${K_{ab}}^c = \frac{1}{n-2} q_{ab} K^c + {C_{ab}}^c \,, \tag{3}$$

$K^c = g^{ab} {K_{ab}}^c$ is called *extrinsic curvature vector* or *mean curvature vector*.

Submanifold

For an arbitrary normal vector X, one can define

$$K^{(X)}_{ab} = -K_{ab}{}^{c} X_c = q_a{}^{c} q_b{}^{d} \nabla_c X_d \,,$$

This is the usual second fundamental tensor along X direction, the expansion and the shear tensor are respectively given by

$$\theta^{(X)} = -K^c X_c \,,$$

$$\sigma^{(X)}_{ab} = -C_{ab}{}^{c} X_c \,.$$

After introducing the covariant derivative on the submanifold and normal covariant derivative, we have generalized Gauss equation, Ricci equation and Codazzi equation:

Submanifold

Gauss equation:

$$R_{abcd} = K_{ca}{}^{e} K_{bde} - K_{cb}{}^{e} K_{ade} + q_a{}^{e} q_b{}^{f} q_c{}^{g} q_d{}^{h} \, \mathscr{R}_{efgh} \,, \qquad (4)$$

Ricci equation:

$$\Omega_{abcd} = q_a{}^{e} q_b{}^{f} h_c{}^{g} h_d{}^{h} \mathscr{R}_{efgh} + K_{aed} K_b{}^{e}{}_{c} - K_{bed} K_a{}^{e}{}_{c} \,. \qquad (5)$$

Codazzi equation:

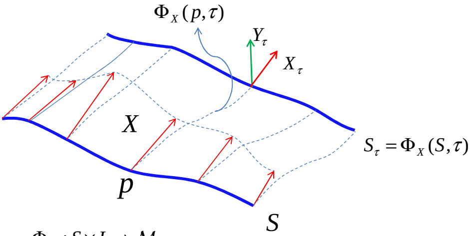
$$\tilde{D}_a K_{bcd} - \tilde{D}_b K_{acd} = -q_a{}^{e} q_b{}^{f} q_c{}^{h} h_d{}^{g} \mathscr{R}_{efhg} \,. \qquad (6)$$

For an arbitrary normal vector Y, it gives

$$\left(\frac{n-3}{n-2}\right) D_a \theta^{(Y)} - D_b \sigma_a^{(Y)b} + K_d \tilde{D}_a Y^d - K_a{}^{b}{}_{d} \tilde{D}_b Y^d = q_a{}^{e} q^{bc} Y^d \mathscr{R}_{ebcd} \,. \qquad (7)$$

The deformation defined by Andersson et al

L. Andersson, M. Mars and W. Simon, Phys. Rev. Lett. 95, 111102 (2005); Adv. Theor. Math. Phys. 12, 853 (2008).



$$\Phi_X : S \times I \to \mathcal{M}$$

$$\delta_X \theta^{(Y)} \equiv \partial_\tau \theta^{(Y_\tau)}\Big|_{\tau=0}$$

Our definition of the deformation

Anderssson et. al. Claimed:

The deformation operator δ_X is different from usual Lie derivative $\mathcal{L}_X$.

Our calculation shows: The difference of δ_X and the usual Lie derivative is nothing but a constraint:

$$\mathcal{L}_X({q_a}^b) = 0\,.$$

So our deformation is just the usual Lie derivative with above constraint.

With this consideration, we can get the deformation equation without introducing any local frame.

Deformation equation with an arbitrary codimension

After some calculation, we have

$$\begin{aligned}\mathcal{L}_X K^{(Y)}_{ab} &= q_a{}^c q_b{}^d X^e Y^f \mathscr{R}_{ecdf} + K^{(Y)c}_a K^{(X)}_{bc} - Y^c \tilde{D}_a \tilde{D}_b X_c \\ &\quad + K_{acb}\left(Y_d \tilde{D}^c X^d\right) - K_{abc}\left(X^d \nabla_d Y^c\right) . \qquad (8)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_X \theta^{(Y)} &= q^{cd} X^e Y^f \mathscr{R}_{ecdf} - K^{(Y)ab} K^{(X)}_{ab} \\ &\quad - Y^c \tilde{D}_a \tilde{D}^a X_c - K_c \left(X^d \nabla_d Y^c\right) . \qquad (9)\end{aligned}$$

For a tangent vector ϕ^a, the Lie derivative of $\theta^{(Y)}$ along ϕ^a is constrained by the Codazzi equations (6) and (7):

$$\left(\frac{n-3}{n-2}\right)\mathcal{L}_\phi \theta^{(Y)} = \phi^a D_b \sigma^{(Y)b}_a - \left(\frac{n-3}{n-2}\right)\phi^a K_d \tilde{D}_a Y^d + \phi^a C_a{}^b{}_d \tilde{D}_b Y^d + q^{fg}\phi^e Y^h \mathscr{R}_{efgh} . \qquad (10)$$

Codimension-1

We can set $h_{ab} = -u_a u_b$, where u^a is an unit timelike normal vector of the hypersurface. So the extrinsic curvature is simply given by $K_{abc} = K_{ab} u_c$. In this case, X is just the evolution vector $X_a = N u_a$ with lapse function N. By selecting $Y_a = u_a$, then

$$\theta^{(Y)} = K = -K^a u_a,$$

and we have

$$-\frac{1}{N}\mathcal{L}_X K_{ab} = -q_a{}^c q_b{}^d \mathscr{R}_{cd} + R_{ab} + K K_{ab} - 2K_{ac}K_b{}^c - \frac{1}{N} D_a D_b N$$

and

$$-\frac{1}{N}\mathcal{L}_X K = \mathscr{R}_{ab} u^a u^b + K^{ab} K_{ab} - \frac{1}{N} D^a D_a N\,.$$

These are just the evolution equations of the hypersurface in Einstein gravity theory.

Codimension-2

From the Gauss equation (4), we find that eq.(9) becomes

$$\begin{aligned}\mathcal{L}_X\theta^{(Y)} &= -\left(\mathscr{G}_{ab}+K_{cda}K^{cd}{}_b\right)\left[X^aY^b-h^{ab}\left(X_eY^e\right)\right] \\ &\quad +\frac{1}{2}\left(R-K_{abc}K^{abc}-K_cK^c\right)\cdot\left(X_eY^e\right) \\ &\quad -Y^e\tilde{D}_c\tilde{D}^cX_e-K_c\left(X^e\nabla_eY^c\right)\,. \qquad (11)\end{aligned}$$

Here:

- $\mathscr{G}_{ab}$ is the Einstein tensor of the spacetime
- R is the scalar curvature of the codimension-2 surface
- $\tilde{D}_a$ is the normal covariant derivative

Codimension-2

By introducing two null vector fields ℓ and n

$$h_{ab} = -\ell_a n_b - n_a \ell_b = \varepsilon_{IJ} e^I_a e^J_b \,, \tag{12}$$

we can define two important quantities:
The first one is:

$$\omega_a = -q_a{}^e n_d \nabla_e \ell^d \,. \tag{13}$$

It is the $SO(1,1)$ connection of the $SO(1,1)$ normal bundle.
The second one is:

$$\kappa_X = -n^c X^e \nabla_e \ell_c \,, \tag{14}$$

which has close relation to the "surface gravity" if horizons are involved.

Focusing and cross focusing equations

By setting $Y_a = \ell_a$ and $X_a = A\ell_a - Bn_a$, then eq.(11) gives result

$$\begin{aligned} \mathcal{L}_X\theta^{(\ell)} &= \kappa_X\theta^{(\ell)} - D_cD^cB + 2\omega^cD_cB \\ &-B\Big[\omega_c\omega^c - D_c\omega^c + \mathscr{G}_{ab}\ell^a n^b - \frac{1}{2}R - \theta^{(\ell)}\theta^{(n)}\Big] \\ &-A\Big[\mathscr{G}_{ab}\ell^a\ell^b + \sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} + \frac{1}{n-2}\theta^{(\ell)}\theta^{(\ell)}\Big]. \end{aligned} \tag{15}$$

In the case where $A = 1, B = 0$, eq.(15) just the so called focusing equation.
In the case where $A = 0, B = -1$, this result gives the cross focusing equation.

Y is dual to X

In this case, we have

$$\kappa_X \theta^{(X)} = \mathscr{G}_{ab} X^a Y^b + \sigma^{(X)}_{ab} \sigma^{(Y)ab} + \frac{1}{n-2} \theta^{(X)} \theta^{(Y)}$$
$$+ D_e (A D^e B - B D^c A - 2AB\omega^e) + A \mathcal{L}_X \theta^{(\ell)} + B \mathcal{L}_X \theta^{(n)} , \tag{16}$$

and

$$\int \kappa_X \mathcal{L}_X \epsilon_q = \int \epsilon_q \left[\mathscr{G}_{ab} X^a Y^b + \sigma^{(X)}_{ab} \sigma^{(Y)ab} + \frac{1}{n-2} \theta^{(X)} \theta^{(Y)} \right]$$
$$+ \int \epsilon_q \left[A \mathcal{L}_X \theta^{(\ell)} + B \mathcal{L}_X \theta^{(n)} \right] , \tag{17}$$

This equation has close relation to the Clausius like equation of the thermodynamics of the quasi-local horizon.

Damour-Navier-Stokes like Equation

From the definition of ω_a in eq.(13), it's not hard to find

$$\mathcal{L}_X \omega_a = K_a{}^b{}_c \tilde{D}_b(\epsilon^{cd} X_d) + D_a \kappa_X - \frac{1}{2} q_a{}^b X^d \epsilon^{ce} \mathscr{R}_{dbce} \,. \tag{18}$$

By using the Codazzi equation (7), we have

$$\mathcal{L}_X \omega_a = D_a \kappa_X + \left(\frac{n-3}{n-2}\right) D_a \theta^{(Y)} - D_c \sigma^{(Y)c}_a + K_c \tilde{D}_a Y^c + q_a{}^b Y^c \mathscr{G}_{bc} \,, \tag{19}$$

where $Y_a = \epsilon_{ab} X^b$.

In the case where X is self-dual or anti-self-dual, i.e., $X = \pm Y$, by considering the Einstein equation, this equation is a kind of *Damour-Navier-Stokes equation*.

Here, we have not introduce any timelike hypersurface or stretched horizon .

Damour-Navier-Stokes like Equation

Let ϕ^a be a tangent vector which satisfies $\mathcal{L}_X \phi^a = 0$ and $D_a \phi^a = 0$, then we get

$$\mathcal{L}_X \int \epsilon_q \left(\phi^a \omega_a\right) = \int \epsilon_q \Bigg\{ \frac{1}{2} \left(D^a \phi^b + D^b \phi^a\right) \sigma_{ab}^{(Y)} + \phi^a Y^b \mathscr{G}_{ab} + A\phi^a D_a \theta^{(\ell)} + B\phi^a D_a \theta^{(n)} \Bigg\} . \quad (20)$$

The angular momentum can be defined as

$$J_\phi = \int \epsilon_q (\phi^a \omega_a) .$$

So, from Damour-Navier-Stokes equation, we can get the deformation equation of angular momentum.

Trapping horizon

- The codimension-2 spacelike surface with $\theta^{(\ell)}\theta^{(n)} = 0$ is called marginal trapped surface.
- The surface with $\theta^{(\ell)}\theta^{(n)} > 0$ is called trapped, and $\theta^{(\ell)}\theta^{(n)} < 0$ is called untrapped.
- A trapped (untrapped) region is the union of all trapped (untrapped) surfaces.

We can give similar definitions by using the extrinsic curvature vector K^a from the relation

$$K^c K_c = -2\theta^{(\ell)}\theta^{(n)} \, .$$

Trapping horizon

- A marginal trapped surface is called *future* if $\theta^{(\ell)} = 0$, $\theta^{(n)} < 0$.
  - (i). if $\mathcal{L}_n \theta^{(\ell)} < 0$, we call the future marginal trapped surface is *outer*.
  - (ii). if $\mathcal{L}_n \theta^{(\ell)} > 0$, the future marginal trapped surface is called *inner*.
- The *past* marginal trapped surface is defined by $\theta^{(n)} = 0$, $\theta^{(\ell)} > 0$.
  - (i). The past marginal trapped surface with $\mathcal{L}_\ell \theta^{(n)} > 0$ is called outer.
  - (ii). The past marginal trapped surface with $\mathcal{L}_\ell \theta^{(n)} < 0$ is called inner.

The so called *trapping horizon* is the closure of a hypersurface foliated by the marginal trapped surfaces[Hayward,1994].

The classification of the trapping horizon inherits from the classification of the marginal trapped surfaces.

Evolution vector

The trapping horizon is foliated by marginal trapped surfaces S_τ. Here τ is called the foliation parameter of the tapping horizon. Assume X is the so called "evolution" vector, i.e., the vector which is tangent to $\mathcal{H}$ and normal to S_τ and satisfies $\mathcal{L}_X \tau = 1$.

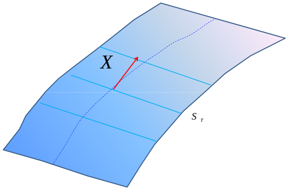


Figure: The evolution vector on trapping horizon

The method with quasi-local energy: spherically symmetric case

For generally spherically symmetric spacetime

$$g = \beta_{\mu\nu}(y) dy^\mu dy^\nu + r(y)^2 \gamma_{ij}(z) dz^i dz^j \,, \tag{21}$$

We have generalized Misner-Sharp energy:

$$\mathscr{E} = \frac{(n-2)\Omega_{n-2}}{16\pi G} r^{n-3} \left(1 - \nabla_a r \nabla^a r\right) \,, \tag{22}$$

By defining

$$\psi_a = \mathscr{T}_{ab} \nabla^b r + w \nabla_a r \,, \qquad w = -\frac{1}{2} h^{ab} \mathscr{T}_{ab} \,, \tag{23}$$

we get

$$\mathcal{L}_X \mathscr{E} = \mathscr{A} \psi_a X^a + w \mathcal{L}_X \mathscr{V} \,, \tag{24}$$

The method with quasi-local energy: spherically symmetric case

By selecting X to be the evolution vector on the trapping horizon , on the trapping horizon, we have

$$\mathscr{A}\psi_a X^a = \left(\frac{\kappa}{2\pi}\right)\mathcal{L}_X S\,, \tag{25}$$

The surface gravity is defined as

$$\frac{\kappa}{2\pi} = \frac{4G}{n-2}\left[\left(\frac{n-3}{\Omega_{n-2}}\right)\frac{\mathscr{E}}{r^{n-2}} - wr\right]\,. \tag{26}$$

The evolution of $\mathscr{E}$ on the trapping horizon becomes

$$\mathcal{L}_X\mathscr{E} = \left(\frac{\kappa}{2\pi}\right)\mathcal{L}_X S + w\mathcal{L}_X\mathscr{V}\,. \tag{27}$$

This is a first law like equation.

The method with quasi-local energy: More general cases

For general case $g_{ab} = h_{ab} + q_{ab}$ the Ricci tensor of q_{ab} is trace free, we can define a generalized energy

$$\mathscr{E} = \frac{\left(\int \epsilon_q\right)^{\frac{n-3}{n-2}}}{16\pi G (\Omega_{n-2})^{\frac{1}{n-2}}(n-3)} \left\{ \frac{\int \epsilon_q R}{\left(\int \epsilon_q\right)^{\frac{n-4}{n-2}}} - \left(\frac{n-3}{n-2}\right) \frac{\int \epsilon_q K_c K^c}{\left(\int \epsilon_q\right)^{\frac{n-4}{n-2}}} \right\}. \quad (28)$$

- for $n = 4$, this energy reduces to usual four dimension Hawking energy (mass).
- In spherically symmetric case, this energy reduces to Misner-Sharp energy ($n \geq 4$).
- In higher dimension, we only consider the case where q_{ab} is Einstein.

The method with quasi-local energy: general cases

The deformation of the energy is given by

$$\mathcal{L}_X \mathscr{E} = \left(\frac{n-3}{n-2}\right)\left(\frac{\mathscr{E}}{\mathscr{A}}\right)\mathcal{L}_X \mathscr{A} + \mathscr{A}^{\frac{n-3}{n-2}}\mathcal{L}_X\left(\frac{\mathscr{E}}{\mathscr{A}^{\frac{n-3}{n-2}}} - \mathscr{K}\right) \quad (29)$$

where $\mathcal{L}_X \mathscr{K} = 0$ with

$$\mathscr{K} = \frac{1}{16\pi G\left(\Omega_{n-2}\right)^{\frac{1}{n-2}}(n-3)}\left(\frac{\int \epsilon_q R}{\left(\int \epsilon_q\right)^{\frac{n-4}{n-2}}}\right). \quad (30)$$

On the horizon, by selecting X to be the evolution vector of the horizon, we have

$$\mathcal{L}_X \mathscr{E} = \left(\frac{n-3}{n-2}\right)\left(\frac{\mathscr{E}}{\mathscr{A}}\right)\mathcal{L}_X \mathscr{A}\,. \quad (31)$$

The method with quasi-local energy: general cases

More detailed, the deformation of the energy is given by

$$\mathcal{L}_X \mathscr{E} = \frac{1}{8\pi G}\left(\frac{L}{n-2}\right)\int \epsilon_q \Bigg\{ - K^e \tilde{D}_c \tilde{D}^c X_e$$

$$- \left(\mathscr{G}_{ab} + C_{cda} C^{cd}{}_b\right)\left[K^a X^b - \frac{1}{2} h^{ab}\left(K_e X^e\right)\right]$$

$$+ \frac{1}{2}\left(\mathscr{G}_{ab} h^{ab}\right) \cdot \left(K_e X^e\right)\Bigg\}, \tag{32}$$

where $L = \mathscr{A}^{\frac{1}{n-2}} / \left(\Omega_{n-2}\right)^{\frac{1}{n-2}}$,

The method with quasi-local energy: general cases

If we introduce null frames, generally, the evolution of the energy on the trapping horizon is given by

$$\mathcal{L}_X \mathscr{E} = \int \epsilon_q \left[\alpha \left(\mathscr{T}_{ab} \ell^a \ell^b + \frac{1}{8\pi G} \sigma^{(\ell)}_{ab} \sigma^{(\ell)ab} \right) + \beta \left(\mathscr{T}_{ab} \ell^a n^b + \frac{\zeta_a \zeta^a}{8\pi G} \right) \right]. \tag{33}$$

where α and β are determined by the components of X, and ζ has close relation to the SO(1,1) connection ω_a.

- The contribution of the usual matter fields — $\mathscr{T}_{ab}\ell^a\ell^a$ and $\mathscr{T}_{ab}\ell^a n^a$;
- The contribution of the gravitational radiation — $\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab}$ and $\zeta_a\zeta^a$.

Method without quasi-local energy: equilibrium state

Equilibrium state: Null trapping horizon.

The evolution vector X is null. From focusing and cross focussing equations, we find: On the null future trapping horizon, we have

$$\sigma_{ab}^{(\ell)} = 0 \,, \qquad \mathscr{G}_{ab}\ell^a\ell^b = 0 \,, \tag{34}$$

and on the null past trapping horizon, we have

$$\sigma_{ab}^{(n)} = 0 \,, \qquad \mathscr{G}_{ab}n^a n^b = 0 \,. \tag{35}$$

$\mathscr{G}_{ab}\ell^a\ell^b = 0$ and $\mathscr{G}_{ab}n^a n^b = 0$ just imply that there are no matter flux across the codimension-2 surface. $\sigma_{ab}^{(\ell)} = 0$ and $\sigma_{ab}^{(n)} = 0$ means that there are no gravitational radiation across the codimension-2 surface.

Method without quasi-local energy: equilibrium state

From Damour-Navier-Stokes equation, we find

$$\mathcal{L}_X \omega_a - D_a \kappa_X = 0\,. \tag{36}$$

if one requires that ω_a does not evolve, i.e., $\mathcal{L}_X \omega_a = 0$, then, from above equation, one gets $D_a \kappa_X = 0$ on the codimension-2 surface.

Furthermore, if $\mathcal{L}_X \kappa_X = 0$ is required, then κ_X is a constant on the null trapping horizon.

In these null cases, we have

$$X^a \nabla_a X^b = \pm \kappa_X X^b\,, \tag{37}$$

Method without quasi-local energy: equilibrium state

Conclusively, on these null trapping horizons, there are no gravitational radiation and matter flux, and κ_X's are constants. These properties correspond to the equilibrium state of the thermodynamics of the horizon.

Further, eqs.(17) and (20) just mean

$$\left(\frac{\kappa_X}{2\pi}\right)\mathcal{L}_X S = 0\,, \qquad \mathcal{L}_X J_\phi = 0\,, \tag{38}$$

where $S \sim \int \epsilon_q$ and $J_\phi \sim \int \epsilon_q\,(\phi^a \omega_a)$ can be explained as the entropy and the angular momentum associated with the null trapping horizons.

Method without quasi-local energy: Near equilibrium state

The near equilibrium means that X is almost a null vector.

$$X^a = \ell^a - Cn^a \,, \tag{39}$$

For the future trapping horizon, Booth *et.el.*(2003) give three *slowly expanding conditions* :

(F-i). The so called evolving parameter $\epsilon \ll 1$ with

$$\frac{\epsilon^2}{L^2} = \max\left[|C|\left(\|\sigma^{(n)}\|^2 + (8\pi G)\mathscr{T}_{ab}n^a n^b + \frac{1}{n-2}\theta^{(n)}\theta^{(n)}\right)\right]; \tag{40}$$

(F-ii). The Ricci scalar, the $SO(1,1)$ normal connection and the energy-momentum tensor satisfy

$$|R|\,, \quad \|\omega_a\|^2 \quad \text{and} \quad (8\pi G)\mathscr{T}_{ab}\ell^a n^b \preceq \frac{1}{L^2}\,;$$

Method without quasi-local energy: Near equilibrium state

(F-i). The derivatives of horizon fields are at most the same order in ϵ as the (maximum of the) original fields. For example,

$$\|D_a C\| \preceq \frac{C_m}{L}\,, \qquad \|D_a D_b C\| \preceq \frac{C_m}{L^2}\,.$$

Here, $\|\cdot\|$ is the norm of (tangent) tensor fields on the codimension-2 Riemannian manifold, while $|\cdot|$ is the absolute value of some scalar. The quantity L is some length scale of the codimension-2 surface. For example, the radius of the closed $(n-2)$ manifold: $L = (\mathscr{A}/\Omega_{n-2})^{\frac{1}{n-2}}$ which has been defined just bellow eq.(32). C_m is the maximum value of $|C|$ on the codimension-2 surface. The relation $E \preceq F$ means $E \leq k_0 F$ for some constant k_0 of order one.

Method without quasi-local energy: Near equilibrium state

The slowly evolving parameter ϵ defined in the condition (F-i) is independent of the relabeling of the foliation and the rescaling of the null frame.

Remembering in the case of future null trapping horizon, to ensure that some physical quantities do not evolve, we have required the condition $\mathcal{L}_X \omega_a = 0$ and $\mathcal{L}_X \kappa_X = 0$. These just mean that ω_a and κ_X do not evolve respect to the evolution vector X^a. Similarly, here there are also *slowly evolving conditions* :

(F-i'). $\|\mathcal{L}_X \omega_a\|$ and $|\mathcal{L}_X \kappa_X| \preceq \epsilon / L^2$;

(F-ii'). $|\mathcal{L}_X \theta^{(n)}| \preceq \epsilon / L^2$.

Method without quasi-local energy: Near equilibrium state

For the past trapping horizon, we can gives similar conditions to describe the slowly expanding properties:

(P-i). The evolving parameter $\epsilon \ll 1$ with

$$\frac{\epsilon^2}{L^2} = \max\left[|C|\left(\|\sigma^{(\ell)}\|^2 + (8\pi G)\mathscr{T}_{ab}\ell^a\ell^b + \frac{1}{n-2}\theta^{(\ell)}\theta^{(\ell)}\right)\right];$$

(P-ii). The Ricci scalar, the $SO(1,1)$ normal connection and the energy-momentum tensor satisfy

$$|R|\,, \quad \|\omega_a\|^2 \quad \text{and} \quad (8\pi G)\mathscr{T}_{ab}\ell^a n^b \preceq \frac{1}{L^2}\,;$$

(P-iii). The derivatives of horizon fields are at most the same order in ϵ as the (maximum of the) original fields. For example,

$$\|D_a C\| \preceq \frac{C_m}{L}\,, \qquad \|D_a D_b C\| \preceq \frac{C_m}{L^2}\,.$$

Method without quasi-local energy: Near equilibrium state

The slowly evolving conditions are given:

(P-i'). $\|\mathcal{L}_X \omega_a\|$ and $|\mathcal{L}_X \kappa_X| \preceq \epsilon/L^2$;

(P-ii'). $|\mathcal{L}_X \theta^{(\ell)}| \preceq \epsilon/L^2$.

With these conditions, one can find that κ_X is nearly a constant on the past trapping horizon. So it can also be expanded as

$$\kappa_X = \kappa_o + \mathscr{O}(\epsilon)\,.$$

Method without quasi-local energy: Near equilibrium state

Clausius like equations:

For the future slowly evolving trapping horizon

$$\left(\frac{\kappa_o}{8\pi G}\right)\mathcal{L}_X\mathscr{A} = \int \epsilon_q \left[\mathscr{T}_{ab}\ell^a\ell^b + \sigma^{(\ell)}_{ab}\sigma^{(\ell)ab}\right] , \tag{41}$$

Similarly, for the past slowly evolving horizon, we have

$$-\left(\frac{\kappa_o}{8\pi G}\right)\mathcal{L}_X\mathscr{A} = \int \epsilon_q \left[\mathscr{T}_{ab}n^a n^b + \sigma^{(n)}_{ab}\sigma^{(n)ab}\right] . \tag{42}$$

Geometry of FRW universe

The metric of the FLRW universe $(\mathcal{M}, g)$ is

$$g = -dt^2 + \frac{a^2}{1-kr^2}dr^2 + a^2r^2d\Omega_{n-2}^2\,, \tag{43}$$

by introducing two null vectors ℓ and n

$$\ell_a dx^a = \sqrt{\frac{1}{2}}\left(-dt + \frac{a}{\sqrt{1-kr^2}}dr\right)\,, \tag{44}$$

$$n_a dx^a = \sqrt{\frac{1}{2}}\left(-dt - \frac{a}{\sqrt{1-kr^2}}dr\right)\,. \tag{45}$$

So we have $h_{ab} = -\ell_a n_b - n_a \ell_b$, while q_{ab} is just the metric for the sphere part, i.e.,

$$q_{ab}dx^a dx^b = a^2r^2d\Omega_{n-2}^2.$$

Null trapping horizons

Null trapping horizons exist only when $k = 0$.
Further, only inner horizon exists in the future case, and only outer horizon exists in the past case.
On the null trapping horizons (future and past), the Hubble parameter H is always a constant.

We only consider the past outer case.

Slowly past evolving trapping horizons in FRW universe

The evolution vector X can be expressed as

$$X^a = \alpha \ell^a - n^a \, , \tag{46}$$

where

$$\alpha = \frac{\dot{H}}{\dot{H} + 2H^2} \, . \tag{47}$$

From the definition, the evolving parameter ϵ in the condition (P-i) becomes (we only consider the four dimension case, and choose L to be the radius $\tilde{r} = 1/|H|$ for $k = 0$.)

$$\frac{\epsilon^2}{\tilde{r}^2} = |\alpha| \left(\mathscr{G}_{ab} \ell^a \ell^b + \frac{1}{2} \theta^{(\ell)} \theta^{(\ell)} \right) \, . \tag{48}$$

Straightforward calculation shows: on the trapping horizons, ϵ's are given by

$$\epsilon^2 = |\alpha| \left(4 - \frac{\dot{H}}{H^2}\right) . \tag{49}$$

By defining

$$s = -\frac{\dot{H}}{H^2} > 0 \,, \tag{50}$$

then, from the expression of α in eq.(47), we have

$$\alpha = -\frac{s}{2 - s} \,. \tag{51}$$

The evolution parameter ϵ now has a simple form

$$\epsilon^2 = s \left(\frac{4 + s}{2 - s}\right) . \tag{52}$$

So, the requirement of the evolving parameter $\epsilon \ll 1$ automatically implies that $s = -\dot{H}/H^2$ is very small.

It's not hard to find

$$\mathcal{L}_X \kappa_X = \frac{2H^2 s}{(2-s)^3}\left[2 - s + s^2 + \left(\frac{\ddot{H}}{\dot{H}H}\right)\right]. \qquad (53)$$

So the slowly evolving condition of κ_X requires that $|\ddot{H}/H^3|$ is also a small quantity.

Thermodynamics on slowly evolving trapping horizon in FRW

For the past horizon, from eq.(42), we have

$$-\frac{\kappa_o}{8\pi G}\mathcal{L}_X \mathscr{A} = \int \epsilon_q \mathscr{T}_{ab} n^a n^b \,. \tag{54}$$

Up to second order of ϵ (or the first order of s).

The temperature of the system can be expanded as

$$T = \frac{\kappa_X}{2\pi} \sim \frac{H}{2\pi}\left(1 - \frac{s}{2}\right) + \mathscr{O}(\epsilon^4)\,.$$

Temperature from the formalism with quasi-local energy

The surface gravity κ in eq.(26) becomes

$$\frac{\kappa}{2\pi} = -\frac{H}{2\pi}\left(1 - \frac{s}{2}\right) , \tag{55}$$

where s is defined in eq.(50). So the temperature of the past outer trapping horizon is

$$T = \frac{|\kappa|}{2\pi} = \frac{H}{2\pi}\left(1 - \frac{s}{2}\right)$$

The Clausius relation is

$$A\psi_a X^a = \frac{\kappa}{2\pi}\mathcal{L}_X S \, . \tag{56}$$

Conclusions and Discussions

- General deformation equations are given without introducing any local frames.
- The Hawking energy is generalized into higher dimension. However, this energy is only interesting when the codimension-2 surface is Einstein. Recently Bray proposed an energy form

$$\mathcal{M}(S) = \frac{1}{2}\left(\frac{|S|}{\omega_{n-2}}\right)^{\frac{n-3}{n-2}} - \frac{1}{2(n-2)^2}\left(\frac{1}{\omega_{n-2}}\int_S H^{\frac{2(n-2)}{n-1}} dS\right)^{\frac{n-1}{n-2}} \tag{57}$$

- In cosmology, the condition of the slowly evolving trapping horizon has close relation to the slow-roll condition.

Thanks for your attention!