Universal Property of Quantum Gravity implied by Uniqueness Theorem of Bekenstein-Hawking Entropy

(Universal = Independent from any existing model of quantum gravity)

SAIDA Hiromi (Daido Univ. at Nagoya)

- ◇ Entropy 13 (2011) 1611 , arXiv:1109.0842
- ◇ Invited as a Feature (and refereed) Paper by J.Bekenstein
  for Special Issue "BH Thermodynamics" in a journal "Entropy"

1. Introduction

1.1 Approach to Quantum Gravity (QG) : Not a model building

- **Issues which need QG for complete answer** $\Longleftrightarrow$ A window to QG

  BH thermodyn. , BH evapo. , Info. Paradox , BH/Initial Sing.

  $\rightarrow$ Here we watch BH Thermodyn.

- Aim of this talk :

  Search for a Universal (model independent) Property of QG

1.2 What I want to do : recall the basis of BH Thermodyn.

- General Relativity → Stationary BH is **stable** under metric perturbation

- QFT on stationary BH spacetime

  → **Hawking Rad.** : Black body radiation (BBR)

- **Presuppositions** of BBR :
  - Thermodyn. and Statist. Mach. are right
  - Arbitrary system is in thermal equilibrium
  - The system emits arbitrary quantum field

  → For the emitted quantum field (= the collection of harmonic oscillators)

  [Energy expectation value by Statist. Mech] − [zero point energy]

$$= \frac{\omega}{\exp(\omega/T) \pm 1}$$: **Planck Spectrum** (ω : frequency of mode fn.)

⇓— From these theoretical facts · · ·

$\Downarrow$

- Presuppositions of this talk :

Presup.1 : Stationary BH is a stable thermal equilibrium state of the underlying QG.

Presup.2 : Equilibrium thermodyn. and statist. mech. are applicable to thermal properties of stationary BHs.

$\rightarrow$ Under these presuppositions, QG should have

the property justifying the Boltzmann formula, $S = \ln \Omega$ **,**

which the ordinary quant. mech. of laboratory systems has.

$\rightarrow$ This property is regarded as

the "universal" property of QG $\leftarrow$ The point of this talk

1.3 Concrete issues in following discussion

- Issue on Presup.1 (BH thermodynamics) :

  Boltzmann formula determines entropy **uniquely**, but $\cdots$

  In BH thermodyn., uniqueness of BH entropy has not been proven.

  $\rightarrow$ Give a rigorous proof to **uniqueness theorem of BH entropy**

  in the framework of *axiomatic thermodynamics*

- Issue on Presup.2 (ordinary quant. mech.) :

  For the number of states of ordinary quantum system Ω,

  a thermodyn. limit, $\lim_{V\to\infty} \dfrac{\ln\Omega}{V}$ $(\neq \infty)$, must exist !

  $\rightarrow$ Show **the conditions for interaction pot. of particles**

  which ensure the existence of the thermodyn. limit.

  $\rightarrow$ **Ruelle-Tasaki Thm & Dobrushin Thm** of ordinary quant. mech.

2. Uniqueness Thm of Bekenstein-Hawking Entropy

H.S., Entropy 13 (2011) 1611 , arXiv:1109.0842

◇ $S_{\rm BH}$ **: Bekenstein-Hawking Entropy** $\left(= \frac{1}{4}\cdot 4\pi(2M)^2\right)$

◇ $K_{\rm BH}$ **: Arbitrary quantity satisfying, (1) extensivity, (2) additivity and (3) increase by adiabatic process**

$\Rightarrow$ $K_{\rm BH}$ **is an affine transform of** $S_{\rm BH}$ **:**

$$K_{\rm BH} = \alpha\, S_{\rm BH} + \eta$$

where α (>0) **and** η **are constants.**

Note : By this Thm, any other property is also uniquely determined.

$\Downarrow$

Then it is justified to consider the Boltzmann formula which determines the entropy uniquely !

3. Property Justifying Boltzmann Formula in Quant. Mech.

3.1 Setting

- System : Non-relativistic identical particles $\left\{\begin{array}{l} V: \text{ volume of system} \\ N: \text{ number of particles} \\ m: \text{ mass of one particle} \end{array}\right.$
- Hamiltonian : $H_{V,N} := -\dfrac{1}{2m}\sum_{i=1}^{N} \triangle_i + \Phi$
- Interaction pot.: $\Phi(\vec{x}_1, \cdots, \vec{x}_N) = \sum_{j=1}^{N} \sum_{1 \le i_1 < \cdots < i_j \le N} \phi^{(j)}(\vec{x}_{i_1}, \cdots, \vec{x}_{i_j})$

  ↑ $\Phi = 0$ at "infinity"
- Energy eigen value: $E_k(V, N)$ $\quad k = 1, 2, \cdots$

  → Increasing order : $E_k(V, N) \le E_{k+1}(V, N)$

  → Ground energy : $E_G(V, N) = E_1 = \cdots = E_d$ (d : degeneracy)
- Number of states : $\Omega_{V,N}(U) := \max_{E_k \le U} k$ (# eigen states of $E_k \le U$)

3.2 Ruelle-Tasaki theorem

Ruelle 1969 & 1999 ; Tasaki 2008

Ruelle-Tasaki Thm gives Sufficient conditions for the validity of Boltzmann formula.

- The sufficient conditions given in this theorem

Cond.A : j-particle interaction pot. $\phi^{(j)}$ is negative at "infinity".
i.e. There exists a constant $r_A\,(>0)$ so that

$$\phi^{(j)}(\vec{x}_{i_1},\cdots,\vec{x}_{i_j}) \le 0 \quad \textbf{for} \quad r_A \le \min_{\substack{k\neq l \\ k,l=1,\cdots,j}} \left|\vec{x}_{i_k}-\vec{x}_{i_l}\right|$$

Cond.B : Total interaction pot. Φ is bounded below.
i.e. There exists a constant $\phi_B\,(>0)$ so that

$$\Phi(\vec{x}_1,\cdots,\vec{x}_N) \ge -N\,\phi_B$$

Then $\cdots$

- Results of Ruelle-Tasaki theorem

Quantum system satisfying conditions A and B

has following unique limits :

Result 1 : $\varepsilon_g(\rho) := \lim_{l.s.l.} \dfrac{E_G(V,N)}{V}$ $\left(\lim_{l.s.l.} : V \to \infty \text{ , } \rho := \dfrac{N}{V} = \textbf{const.}\right)$

Result 2 : $\sigma(\rho,\varepsilon) := \lim_{t.l.} \dfrac{\ln \Omega_{V,N}(U)}{V}$ $\left(\varepsilon := \dfrac{U}{V} \geq \varepsilon_g(\rho) \right)$

and σ **is concave w.r.t.** (ρ,ε) **and increasing w.r.t.** ε

$\left(\lim_{t.l.} : V \to \infty \textbf{ and } \rho,\varepsilon = \textbf{consts.}\right)$

Note 1 : Result 1 gives the lower bound of ε for result 2.

Note 2 : "$\sigma(\rho,\varepsilon) =$ const. for *reversible* adiabatic process" is also shown.

$\to$ With regarding this as one part of the entropy principle $\cdots$

$\sigma(\rho,\varepsilon)$ **is usually regarded as the entropy density!** $\to$ $\underline{\; S = \ln \Omega \;}$

3.3 Dobrushin theorem for quantum system

Dobrushin, Teorija Verojatn. i ee Prim.9 (1964) 626

→ This paper seems to be for classical systems.
I extended to quantum system.

Dobrushin Thm gives a Necessary condition
for the existence of thermal equilibrium states.

- Preparation : Grand Partition Function, $\Xi_V := \sum_{N=1}^{\infty} e^{\beta\mu N}\,\mathrm{Tr}\, e^{-\beta H_{V,N}}$

→ Basic equivalent relation

[Ξ_V exists] $\Longleftrightarrow$ [Statistical equilibrium states of system exist]

• Contents of theorem

Supposition 1 : $\Phi = \sum_{1 \le i < j \le N} \phi^{(2)}(\vec{x}_i, \vec{x}_j)$ **(only 2-particle int.)**

Supposition 2 : $\lim_{l.s.l.} \frac{1}{V^N} \int \cdots \int_V d^3x_1 \cdots d^3x_N \, \triangle_1^{q_1} \cdots \triangle_N^{q_N} \, \Phi =$ **finite**

$(\; q_i = 0, 1, 2, \cdots$ **and** $q_1 + \cdots + q_N \neq 0 \;)$

Under these suppositions $\cdots$

If $I_V^{(2)} := \frac{1}{V^2} \iint_V d^3x_1 \, d^3x_2 \; \phi^{(2)}(\vec{x}_1, \vec{x}_2) < 0$

then $\lim_{l.s.l.} \Xi_V \to \infty$ **(statistical equilibrium does not exits)**

$\to$ Contraposition : For the system of 2-particle interaction,

$$[\; \textbf{Thermal equilibrium exists} \;] \Longrightarrow [\; I_V^{(2)} \ge 0 \;]$$

Note on Suppos.2 : Ex. $\phi^{(2)} \propto \frac{1}{|\vec{x}_i - \vec{x}_j|^\alpha}$ $(\; \alpha \ge 0 \;)$ $\to$ a weak constraint

3.4 Case : therm. equil. is possible for a system of $\Phi = \sum \phi^{(2)}$

- On the sufficient conditions given by Ruelle-Tasaki Thm

  { No counter-example to Cond.A and B is found in laboratory.
    No strong reason to violate Cond.A and B for real thermal systems.

  (also for BHs !)

  $\rightarrow$ **Reasonably, we adopt the conditions A and B.**

- By the necessary condition given by Dobrushin Thm : $\phi^{(2)}$ satisfies $I_V^{(2)} \geq 0$

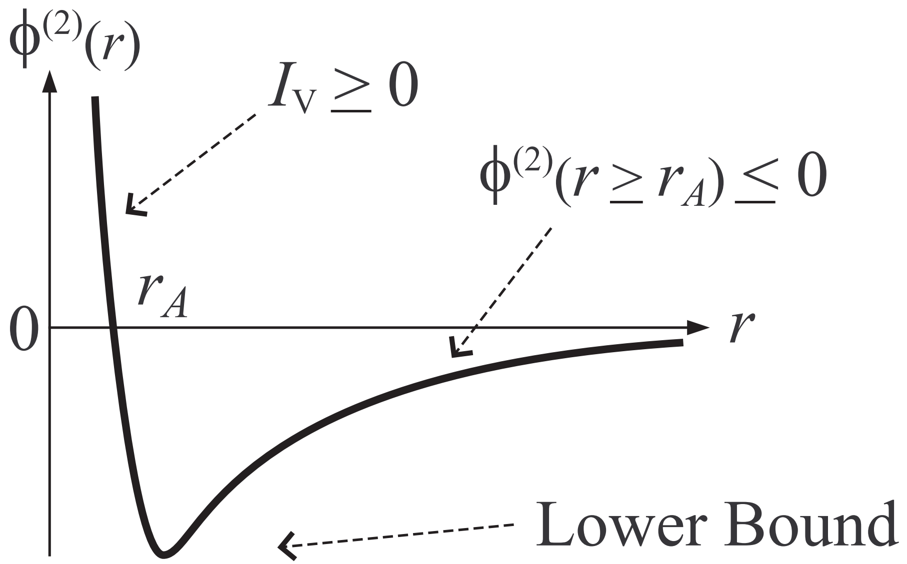
$\Downarrow$

Typical potential is the right one $\longrightarrow$

where we assume for simplicity,

$$\phi^{(2)}(\vec{x}_1, \vec{x}_2) = \phi^{(2)}(r) \ , \ \ r = |\vec{x}_1 - \vec{x}_2|$$

Repulsive at short length scale !



4. Conclusion : Universal Property of Quantum Gravity (QG)

Entropy 13 (2011) 1611 , arXiv:1109.0842

Reasonable Presuppositions based on Uniqueness of BH entropy :

Presup.1 : Stationary BH is a stable thermal equilibrium of QG.

Presup.2 : Thermodyn. and Statist. Mech. are valid for BHs.

$\rightarrow$ QG and ordinary quantum mechanics should share the property which justifies the Boltzmann formula :

Suggestion : Interaction between quantum states of gravity is bounded below and repulsive at short (Planck) length scale.

$\rightarrow$ QG is not necessarily expressed by a potential $\phi^{(2)}(r) \cdots$

Interpretation : Semiclassical Lagrangian of underlying QG should raise a repulsive gravity at Planck length scale.

This property is independent of existing models of QG.

Suppl. Difference of BH thermodyn. and laboratory thermodyn.

- Basic axioms of ordinary thermodyn. are not only 0th to 3rd laws !

  → Ex.: **All state variables are distinguished into two groups, "extensive" and "intensive" variables.**

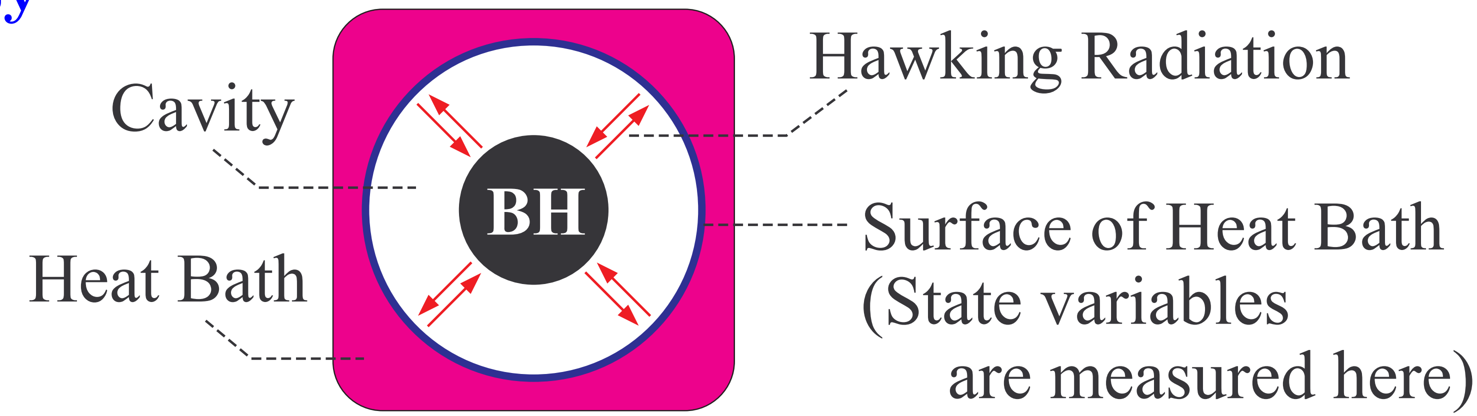
  → This axiom is used in the proof of uniqueness of entropy.

Lieb & Yngvason 1999 ; Tasaki 2000

- How about BH's state variables ? · · · ex : Schwarzschild BH

  → **BH in a cavity of heat bath**

  (by York, 1986)



- Basic scaling and Categorization of BH's state variables

Scaling of basic parameters of system :

$$M \to \lambda M \quad , \quad r_w \to \lambda r_w \quad \text{(length)}$$

Then, BH's state variables are distinguished into 3 groups:

$$\begin{cases} \textbf{Extensive variables } X \textbf{ (e.g. } S_{\rm BH}\textbf{)} & : X \to \lambda^2 X \\ \textbf{Intensive variables } Y \textbf{ (e.g. } T_{\rm BH}\textbf{)} & : Y \to \dfrac{Y}{\lambda} \\ \textbf{Thermodyn. energy } Z \textbf{ (e.g. } F_{\rm BH}\textbf{)} & : Z \to \lambda Z \end{cases}$$

→ This classification is very different from that of ordinary thermodyn.

→ **It is not obvious if BH entropy is unique or not !**

$\cdots$ I have shown the uniqueness.