Beyond delta-N formalism

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The contents of my talk

1. Introduction and Motivation

2. Gradient expansion and delta-N formalism

3. Beyond delta-N formalism

Introduction

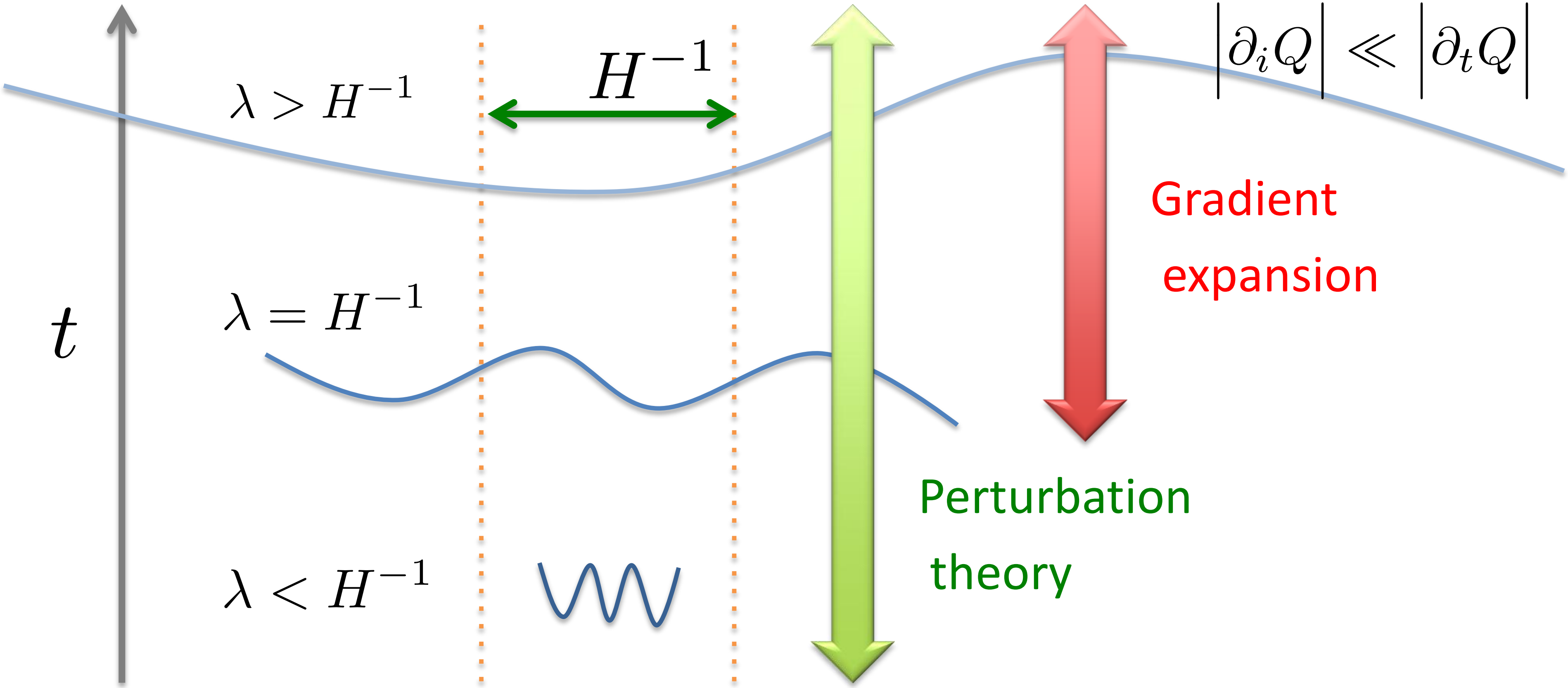
- Inflation is one of the most promising candidates
  as the generation mechanism of primordial fluctuations.

- We have hundreds or thousands of inflation models.
  → we have to discriminate those models

- Non-Gaussianity in CMB will have the key of this puzzle.

- In order to calculate the NG correctly,
  we have to go to the second order perturbation theory,
  but …

Evolution of fluctuation



Concentrating on the evolution of fluctuations on large scales, we don’t necessarily have to solve complicated pertur. Eq.

Gradient expansion approach

- In GE, equations are expanded in powers of spatial gradients.

  → Although it is only applicable to superhorizon evolution, full nonlinear effects are taken into account.

- At the lowest order in GE (neglect all spatial gradients),

  lowest order Eq. = Background Eq.

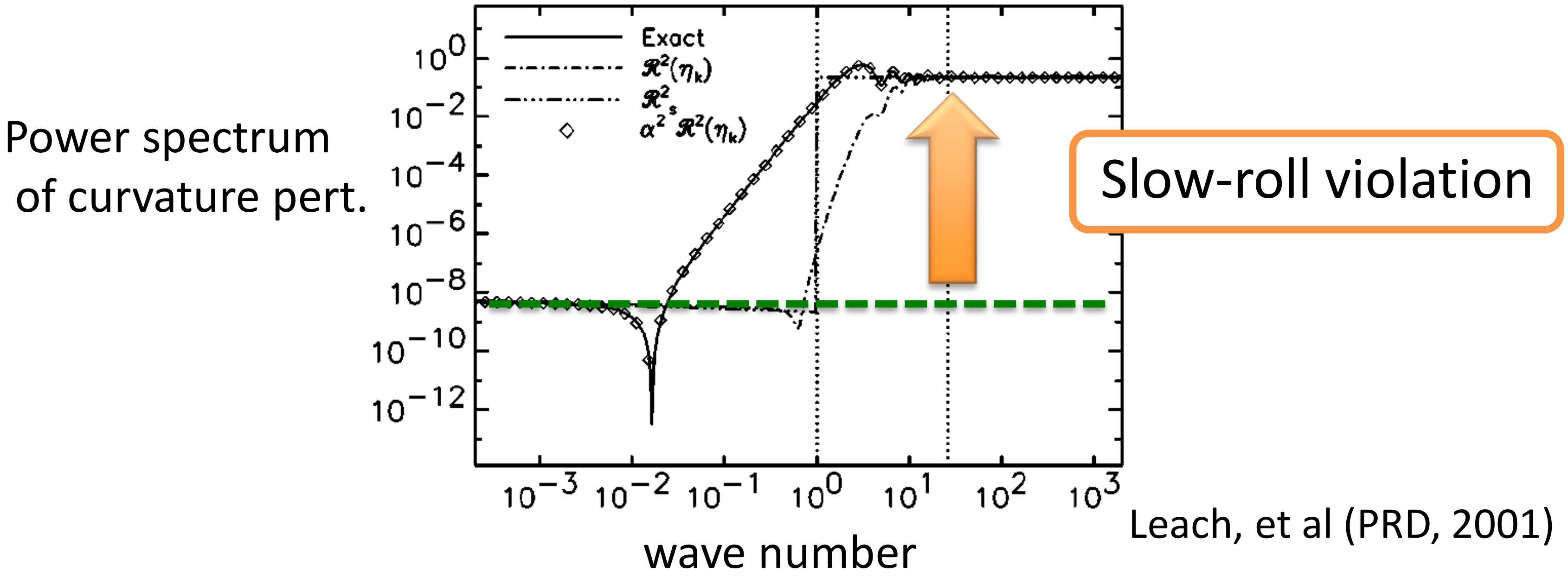
- Just by solving background equations, we can calculate curvature perturbations and NG in them.

  $\mathcal{R} \sim$ (difference of e-fold) : delta-N formalism

- Don’t we have to care about spatial gradient terms ?

Slow-roll violation

- If slow-roll violation occured, we cannot neglect gradient terms.



Leach, et al (PRD, 2001)

- Since slow-roll violation may naturally occur in multi-field inflation models, we have to take into account gradient terms more seriously in multi-field case.

Goal

Our goal is to give the general formalism for solving

the higher order terms in (spatial) gradient expansion,

which can be applied to the case of multi-field.

Gradient expansion approach
and
delta-N formalism

Gradient expansion approach

- On superhorizon scales, gradient expansion will be valid.

$$\left|\partial_i Q\right| \ll \left|\partial_t Q\right| \sim HQ \qquad L \gg H^{-1}$$

$$\partial_i \to \epsilon \; \partial_i$$

→ We expand Equations in powers of spatial gradients : ε

- We express the metric in ADM form

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- We decompose spatial metric g_{ij} and extrinsic curvature K_{ij} into

$$g_{ij} = a^2(t) e^{\boxed{2\Psi}} \gamma_{ij} \qquad \det|\gamma_{ij}| = 1$$

a(t) : fiducial “B.G.”

$$\text{c.f.} \quad \Psi(t_*, 0) = 0$$

Ψ ~ R : curvature perturbation

$$K_{ij}\left(\sim \dot{g}_{ij}\right) = a^2 e^{2\Psi}\left(\frac{1}{3}K\gamma_{ij} + \underline{A_{ij}}\right)$$

traceless

Lowest-order in gradient expansion

- After expanding Einstein equations, lowest-order equations are

lowest-order eq.

$$\frac{1}{3}K^2 = E \qquad -\frac{1}{3}K = \frac{\partial_\tau(ae^\Psi)}{ae^\Psi}$$

$$\frac{1}{\alpha}\partial_t K = \partial_\tau K = \frac{3}{2}(E+P)$$

background eq.

$$3H^2 = \rho_0 \qquad H = \frac{\partial_t a}{a}$$

$$\partial_t(-3H) = \frac{3}{2}(\rho_0 + P_0)$$

→ The structure of lowest-order eq is same as that of B.G. eq with identifications, $d\tau \Leftrightarrow dt$ and $ae^\Psi \Leftrightarrow a$!

lowest-order sol.

$$\phi^{(0)}(\tau) = f(\tau, \phi_i)$$

changing t by τ

background sol.

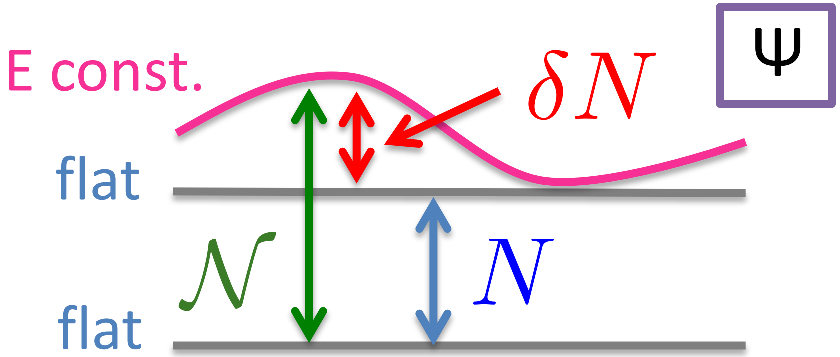
$$\phi_0(t) = f(t, \phi_i)$$

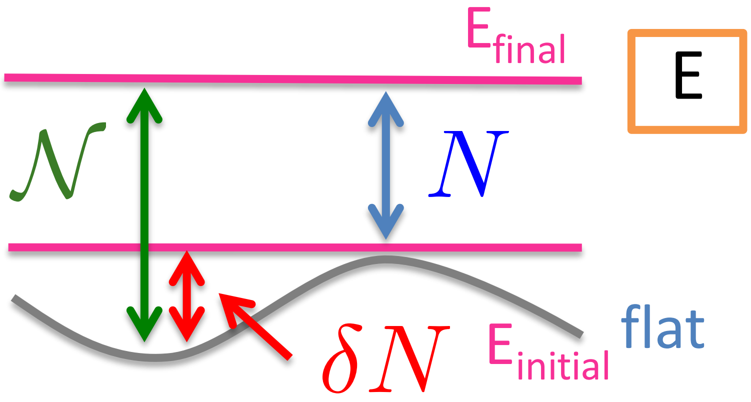
delta-N formalism

- We define the non-linear e-folding number and delta-N.

$$\mathcal{N} \equiv \frac{1}{3}\int K\alpha dt \sim -\int (H+\dot{\Psi})dt \qquad \delta N \equiv \mathcal{N} - N = \Big[\Psi\Big]^{\mathrm{final}}_{\mathrm{initial}}$$

- Choose slicing such that initial : flat & final : uniform energy



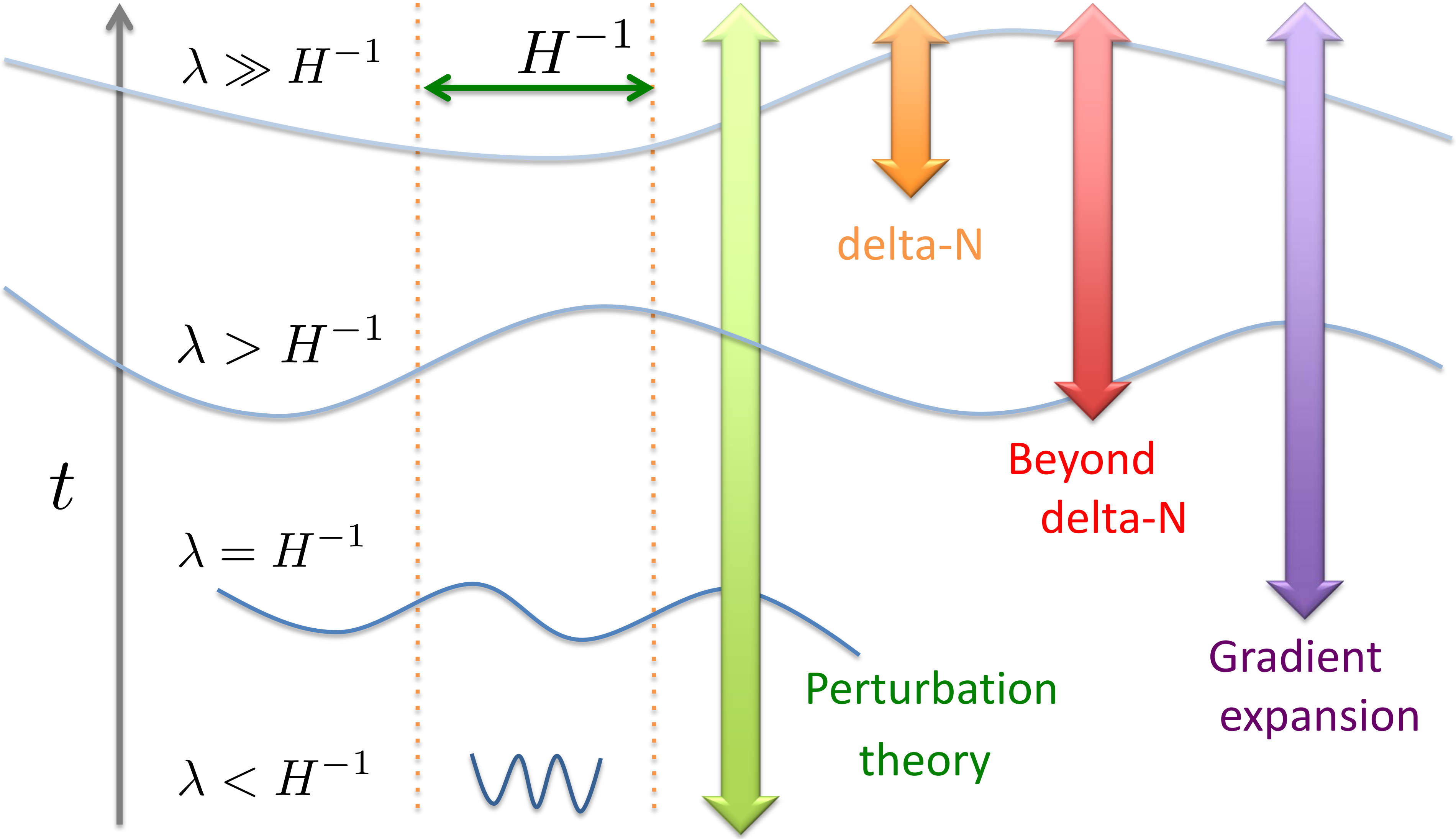


delta-N gives the final curvature perturbation

$$\Psi_E(t_f) = \delta N = N[E_i + \delta E_i, E_f] - N[E_i, E_f]$$

Beyond delta-N formalism

Gradient expansion approach



towards “Beyond delta-N”

- At the next order in gradient expansion,

  we need to evaluate spatial gradient terms.

$$\partial_\tau^2 \phi^{(2)} + 3H\partial_\tau \phi^{(2)} + V_\phi^{(2)} = \triangle \phi^{(0)} \qquad Q^{(2)} = \mathcal{O}(\epsilon^2)$$

- Since those gradient terms are given by the spatial derivative

  of lowest-order solutions, we can easily integrate them…

- Once spatial gradient appeared in equation,

  we cannot use “τ” as time coordinate which depends on x^i

  because integrable condition is not satisfied.

$$d\tau \neq \alpha(t, x^i)dt \quad \Longleftrightarrow \quad dN = H(t)dt$$

→ we cannot freely choose time coordinate (gauge) !!

Beyond delta-N

- We usually use e-folding number (not t) as time coordinate.

$$\mathcal{N} = \frac{1}{3}\int K\alpha dt = -\int (H + \dot{\Psi})dt \to N$$

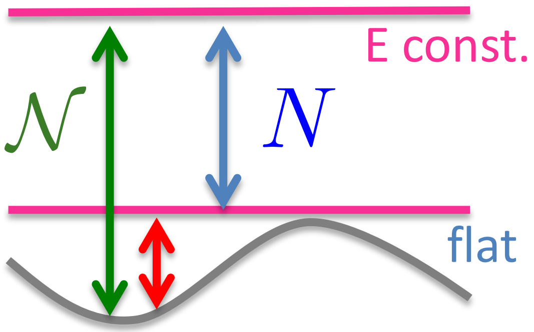
→ We choose uniform N gauge and use N as time coordinate.

- Form the gauge transformation δN : uniform N → uniform E, we can evaluate the curvature perturbation $\Psi_E \sim \delta N$.

lowest order

$$3H^2 = E$$

$$E^{(0)} = E^{(0)}(N, E_i)$$



next order

$$3H^2 = E + \mathcal{O}(\epsilon^2)$$

$$E^{(2)} = E^{(2)}(N, E_i)$$

$$\delta N \sim \Psi_E$$

Summary

- We gave the formalism, “Beyond delta-N formalism”,
  to calculate spatial gradient terms in gradient expansion.

- If you have background solutions, you can calculate
  the correction of “delta-N formalism” with this formalism
  just by calculating the “delta-N”.

Linear perturbation theory

FLRW universe

- For simplicity, we focus on single scalar field inflation.

- Background spacetime : flat FLRW universe

$$ds^2 = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j]$$

Friedmann equation :

$$3\mathcal{H}^2 = a^2\rho_0 = \frac{1}{2}\phi_0'(\eta)^2 + V(\phi_0) \qquad \mathcal{H} \equiv \frac{a'}{a}$$

Linear perturbation

- We define the scalar-type perturbation of metric as

$$ds^2 = a^2\left[-(1+2\alpha)d\eta^2 + 2\triangle^{-\frac{1}{2}}\partial_i\beta d\eta dx^i + \left\{(1+2\mathcal{R})\delta_{ij} + \triangle^{-1}\partial_i\partial_j E\right\}dx^i dx^j\right]$$

(0, 0) : $\triangle\mathcal{R} + \triangle^{\frac{1}{2}}\sigma_g + 3\mathcal{H}(\mathcal{H}\alpha - \mathcal{R}') = -\frac{1}{2}a^2\delta\rho$

$$a^2\delta\rho = \phi_0'\delta\phi' - {\phi_0'}^2\alpha + a^2V_\phi\delta\phi$$

(0, i) : $\mathcal{R}' - \mathcal{H}\alpha = -\frac{1}{2}\phi_0'\delta\phi$

trace : $\mathcal{R}'' + 2\mathcal{H}\mathcal{R}' - \mathcal{H}\alpha' - (2\mathcal{H}' + \mathcal{H}^2)\alpha = -\frac{1}{2}a^2\delta P$

traceless : $(\triangle^{\frac{1}{2}}\sigma_g)' + 2\mathcal{H}(\triangle^{\frac{1}{2}}\sigma_g) = -\triangle(\alpha + \mathcal{R})$

$$\sigma_g \equiv \triangle^{\frac{1}{2}}\beta - E'$$

Linear perturbation : J = 0

- We take the comoving gauge = uniform scalar field gauge.

$$a^2 J = \phi_0' \delta\phi = 0$$

- Combining four equations, we can derive the master equation.

$$\mathcal{R}_c'' + 2\frac{z'}{z}\mathcal{R}_c' - \triangle\mathcal{R}_c = 0 \qquad z \equiv \frac{a\phi_0'}{\mathcal{H}}$$

- On super horizon scales, R_c become constant.

$$\mathcal{R}_c = \mathrm{const.} \qquad \text{and} \qquad \mathcal{R}_c' \propto z^{-2} \sim a^{-2}$$

Einstein equations in J = 0

- Original Einstein equations in J = 0 gauge are

(0, 0) : $$\cancel{\triangle\mathcal{R}} + \triangle^{\frac{1}{2}}\sigma_g + \cancel{3\mathcal{H}(\mathcal{H}\alpha - \mathcal{R}')} = -\frac{1}{2}a^2\delta\rho \boxed{= \frac{1}{2}{\phi_0'}^2\alpha}$$

$$a^2\delta\rho = \phi_0'\delta\phi' - {\phi_0'}^2\alpha + a^2V_\phi\delta\phi$$

(0, i) : $$\mathcal{R}' - \mathcal{H}\alpha = \cancel{-\frac{1}{2}\phi_0'\delta\phi}$$

$$\sigma_g \equiv \triangle^{\frac{1}{2}}\beta - E'$$

trace : $$\mathcal{R}'' + 2\mathcal{H}\mathcal{R}' - \mathcal{H}\alpha' - (2\mathcal{H}' + \mathcal{H}^2)\alpha = -\frac{1}{2}a^2\delta P$$

traceless : $$(\triangle^{\frac{1}{2}}\sigma_g)' + 2\mathcal{H}(\triangle^{\frac{1}{2}}\sigma_g) = \cancel{-\triangle(\alpha + \mathcal{R})}$$

$$\boxed{\mathcal{R}_c' \sim \mathcal{H}\alpha \sim \triangle^{\frac{1}{2}}\sigma_g \rightarrow a^{-2}}$$

$$ds^2 = a^2\left[-(1+2\alpha)d\eta^2 + 2\triangle^{-\frac{1}{2}}\partial_i\beta d\eta dx^i + \left\{(1+2\mathcal{R})\delta_{ij} + \triangle^{-1}\partial_i\partial_j E\right\}dx^i dx^j\right]$$

$R_c = a\,\delta\phi_{\rm flat}$

- We can quantize the perturbation with $u \equiv z\mathcal{R}_c$ $\quad z \equiv \dfrac{a\phi_0'}{\mathcal{H}}$

$$\mathcal{R}_c'' + 2\frac{z'}{z}\mathcal{R}_c' - \triangle\mathcal{R}_c = 0 \quad \Rightarrow \quad u'' - \Big(z''/z + \triangle\Big)u = 0$$

- u is the perturbation of scalar field on R = 0 slice.

$$\mathcal{R}_c \equiv \mathcal{R} + \frac{\mathcal{H}}{\phi_0'}\delta\phi \quad \Rightarrow \quad u = z\mathcal{R}_c = z\mathcal{R} + a\delta\phi = a\delta\phi_{\rm flat}$$

→ quantization is done on flat (R = 0) slice.

→ perturbations at horizon crossing
which give the initial conditions for ∇ expansion
are given by fluctuations on flat slice.

Curvature perturbation ?

- We parameterised the spatial metric as

$$g_{ij} = a^2(t)e^{2\Psi}\gamma_{ij} \xrightarrow{\text{linearlise}} (1+2\Psi)\delta_{ij} + \triangle^{-1}\overset{\text{traceless}}{\underline{\left(\partial_i\partial_j E - \frac{1}{3}\triangle E\delta_{ij}\right)}}$$

- In the linear perturbation, we parametrised the spatial metric as

$$g_{ij} = a^2\left[(1+2\mathcal{R})\delta_{ij} + 2\triangle^{-1}\partial_i\partial_j E\right] \longrightarrow \boxed{\mathcal{R} = \Psi - \frac{1}{3}E}$$

R : curvature perturbation $R^{(3)} = -\frac{4}{a^2}\triangle\mathcal{R}$

- Strictly speaking, Ψ is not the curvature perturbation. $\sigma_g \sim E'$

→ On SH scales, E become constant and we can set E = 0.

→ Ψ can be regarded as curvature perturbation at lowest-order in ∇ expansion. $\Psi \sim \mathcal{R}$

Shear and curvature perturbation

- Once we take into account spatial gradient terms, shear (σ_g or A_{ij}) will be sourced by them and evolve.

  → we have to solve the evolution of E.

$$(\triangle^{\frac{1}{2}}\sigma_g)' + 2\mathcal{H}(\triangle^{\frac{1}{2}}\sigma_g) = -\triangle(\alpha + \mathcal{R})$$

- At the next order in gradient expansion,

  Ψ is given by “delta-N” like calculation.

  In addition, we need to evaluate E.

$$\mathcal{R} = \Psi - \frac{1}{3}E$$

$$\delta N$$

$$E \sim \int dt\sigma_g \sim \int dt\dot{E}$$

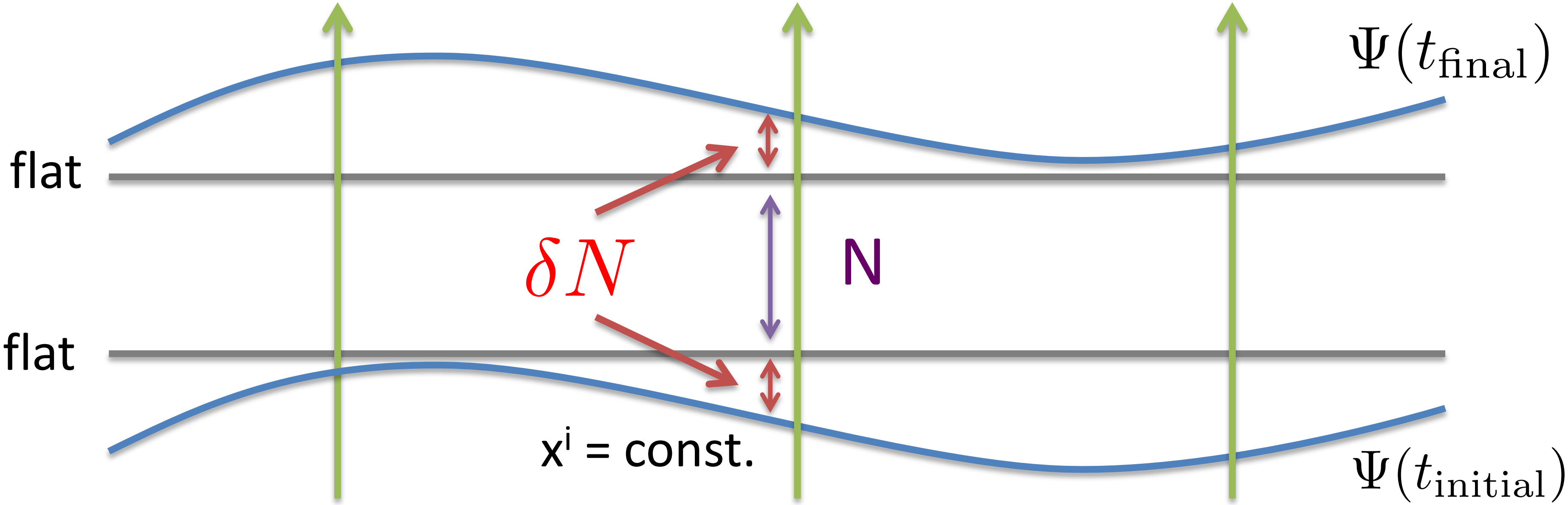
delta-N formalism 1

- We define the non-linear e-folding number

$$\mathcal{N} \equiv \frac{1}{3}\int K\alpha dt \sim -\int (H + \dot{\Psi})dt = N - \Big[\Psi\Big]_{\text{initial}}^{\text{final}}$$

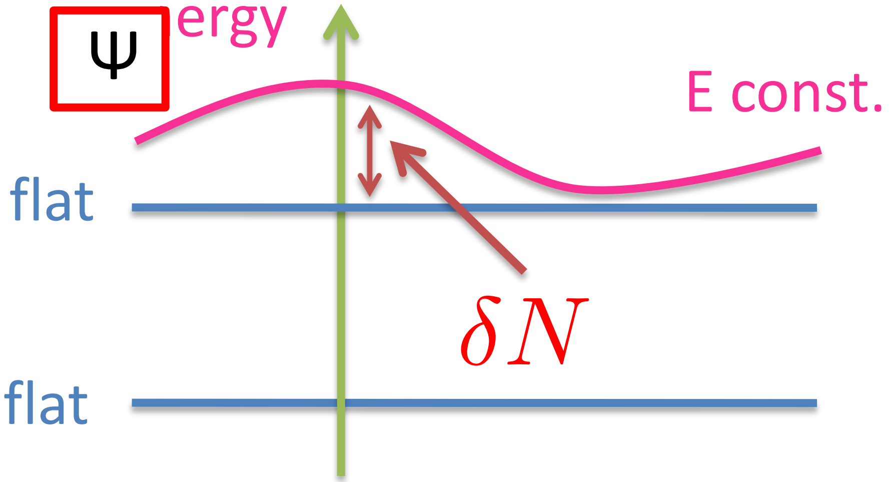
- Curvature perturbation is given by the difference of “N”

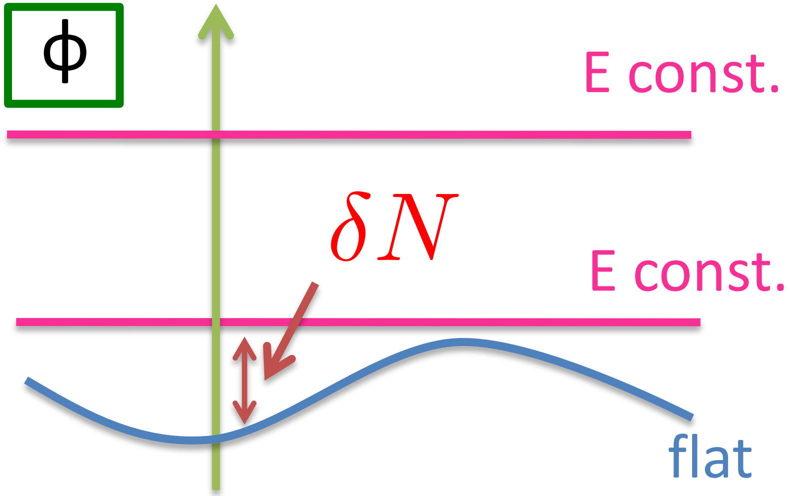
$$\delta N \equiv \mathcal{N} - N = \Psi(t_{\text{final}}) - \Psi(t_{\text{initial}})$$



delta-N formalism 2

- Choose slicing such that initial : flat & final : uniform





delta-N gives the final curvature perturbation

$$\delta N = \Psi_c(t_f) - \cancel{\Psi_f(t_i)} = N[\phi_i + \delta\phi_i, \phi_f] - N[\phi_i, \phi_f]$$

Beyond delta-N

- We usually use e-folding number (not t) as time coordinate.

$$\mathcal{N} = \frac{1}{3}\int K\alpha dt = -\int (H + \dot{\Psi})dt$$

→ We choose uniform N slicing and use N as time coordinate.

- Combining equations, you will get the following equation for ϕ.

$$\frac{V}{3}\left(1 - \frac{1}{6}\partial_N \phi^2\right)^{-1} \partial_N^2 \phi - \frac{V}{9}\partial_N \phi + V_\phi = F^{(2)}\left[N, \phi^{(0)}(N_i), \gamma_{ij}^{(0)}(N_i)\right]$$

$$\boxed{\phi^{(2)} = f\left[N, \phi^{(0)}(N_i), \gamma_{ij}^{(0)}(N_i)\right]}$$

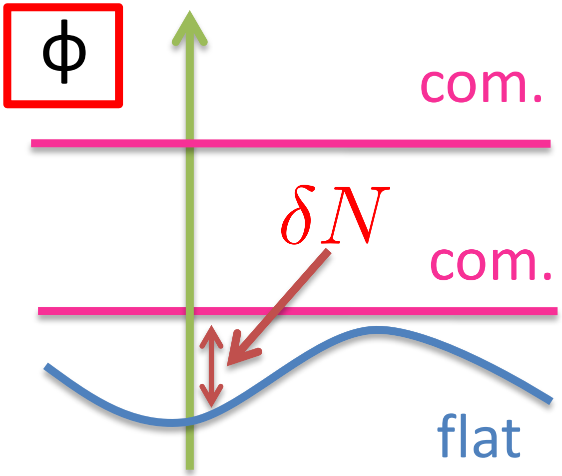
Beyond delta-N 2

- We compute “delta N” from the solution of scalar field.

$$\phi^{(2)}_{\rm final} = F^{(2)}\left[N, \phi^{(0)}(N_i), \gamma^{(0)}_{ij}(N_i)\right]$$

$$\parallel$$

$$\phi^{(2)}_{\rm final} = F^{(2)}\left[N + \delta N, \phi^{(0)} + \delta\phi, \gamma^{(0)}_{ij} + \delta\gamma_{ij}\right]$$



$$\delta N = \Psi^{c}_{\rm final} - \Psi^{\rm flat}_{\rm initial} = \mathcal{R}^{c}_{\rm final} + \frac{1}{3}\underline{\left(E^{c}_{\rm final} - E^{\rm flat}_{\rm initial}\right)}$$

$$\mathcal{R} = \Psi - \frac{1}{3}E$$

$$\partial_N A_{ij} = 3A_{ij} + G^{(2)}\left[N, \phi^{(0)}(N_i), \gamma^{(0)}_{ij}(N_i)\right]$$

Beyond delta-N 3

- We extend the formalism to multi-field case.

- As a final slice, we choose uniform E or uniform K slice
  since we cannot take “comoving slice”. $T^0{}_i$: vectorial

$$K^2 \sim E \qquad \partial_i K = J_i$$ @ lowest order

- We can compute “delta N” form the solution of E, K.

$$K^{(2)} = K^{(2)}\left[N, \phi_I^{(0)}(N_i), \gamma_{ij}^{(0)}(N_i)\right]$$

$$\mathcal{R}^{E,K}_{\rm final} = \delta N - \frac{1}{3}(E^{E,K}_{\rm final} - E^{\rm flat}_{\rm initial})$$

Question

How can we calculate

the correction of delta-N formalism ?

Answer

To calculate the cor. of delta-N,

all you have to do is calculate “delta-N”.