Gradient expansion approach to multi-field inflation

Dept. of Physics, Waseda University

Yuichi Takamizu

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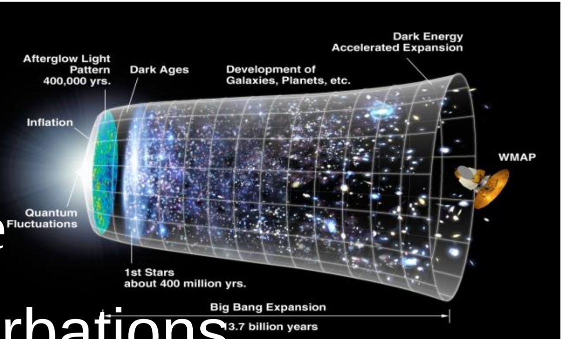
Collaborators: **S.Mukohyama** (IPMU), **J.Yokoyama** (U of Tokyo), **M.Sasaki**, **A.Naruko** (YITP,Kyoto U)

Ref: **PRD 83 043504 (2011)**

JCAP 06 019 (2010) & JCAP 01 013 (2009)

◆ Introduction



● Inflationary scenario in the early universe

- Generation of primordial density perturbations
- **Seeds** of large scale structure of the universe

☐ The primordial fluctuations generated from Inflation are one of *the most interesting prediction of quantum theory of fundamental physics.*

However, we have *known little* information about Inflation

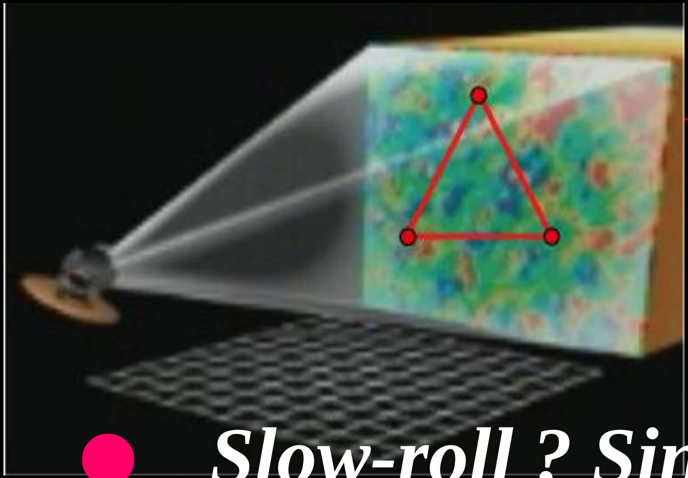
☐ More accurate observations give more information about primordial fluctuations

◆ Non-Gaussianity from Inflation

- PLANCK (2009-)

◆ Non-Gaussianity

$$\zeta(x) = \zeta_G(x) + \frac{3}{5} f_{NL}^{\text{local}} \zeta_G^2(x)$$



● **Current bound**

WMAP 7-year $-10 < f_{\text{NL}}^{\text{local}} < 74$

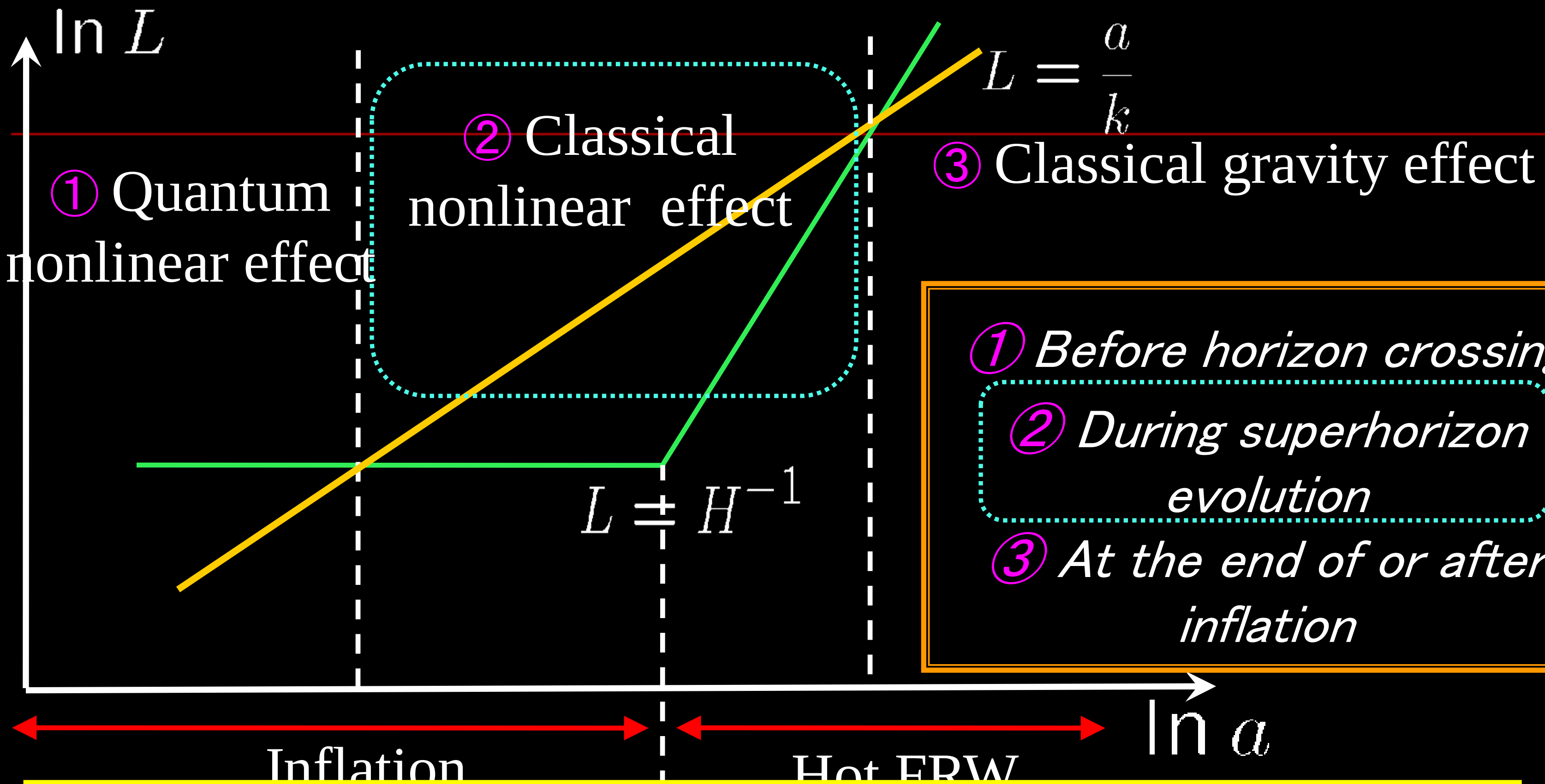
PLANK 2009- detect within $|f_{\text{NL}}| \gtrsim 5$

● ***Slow-roll ? Single field ? Canonical kinetic ?***

- ➢ Standard single slow-roll scalar $f_{\text{NL}} = O(10^{-2})$
- ➢ Many models predicting Large Non-Gaussianity
  （Multi-fields, DBI inflation & Curvaton） $f_{\text{NL}} \gg O(1)$

◻ *Non-Gaussianity will be one of powerful tool to discriminate many possible inflationary models with the future precision observations*

◆ Three stages for generating Non-Gaussianity



① Quantum nonlinear effect

② Classical nonlinear effect

③ Classical gravity effect

① Before horizon crossing

② During superhorizon evolution

③ At the end of or after inflation

◆ Generation of NG in superhorizon needs violation of the condition for ①*slow-roll* or(and) ②*single*

● *Nonlinear perturbations on superhorizon scales*

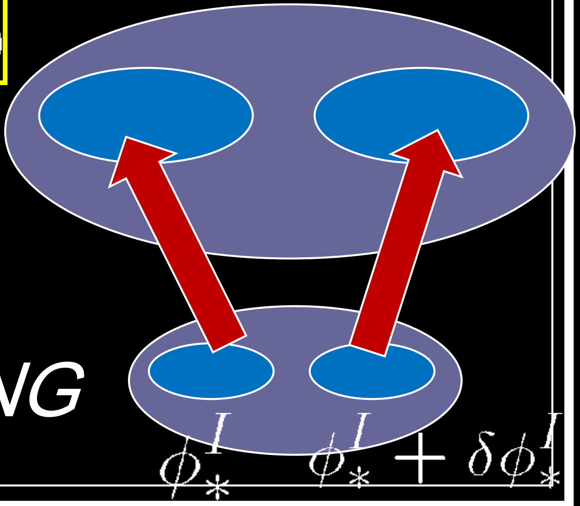
- *Spatial gradient approach* : $\boxed{\epsilon = 1/(HL)}$ Salopek & Bond (90)
  - Spatial derivatives are small compared to time derivative
  - Expand Einstein eqs in terms of small parameter ε, and can solve them for nonlinear perturbations iteratively

(Starobinsky 85,
Nambu & Taruya 96 ,
Sasaki & Stewart 96)

◆ *δ N formalism (Separated universe)*

$$\boxed{\zeta(t,\mathrm{x}) = \delta N \equiv N(t,\mathrm{x}) - N_0(t)} \quad \boxed{O(\epsilon^0)}$$

Curvature perturbation = Fluctuations of
the local e-folding number



◇ *Powerful tool for the estimation of NG*

◆ δ N formalism (Separated universe approach)

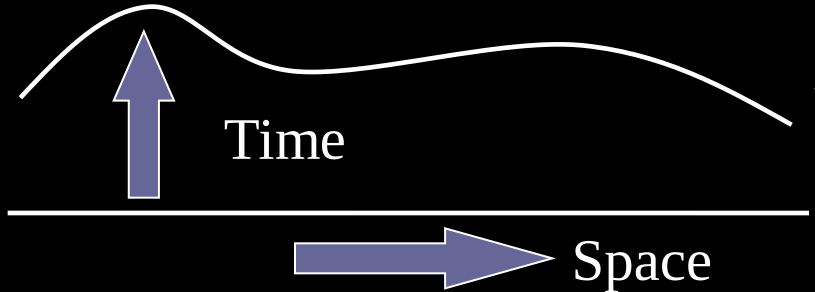
$\det\gamma = 1$

Spatial metric: $g_{ij} = a^2(1+2\zeta)\gamma_{ij} = e^{2N}(1+2\delta N)\gamma_{ij}$

Curvature perturbation e-folding perturbation

$(N = \log a)$

Two hypersurfaces:



Uniform energy density

Time

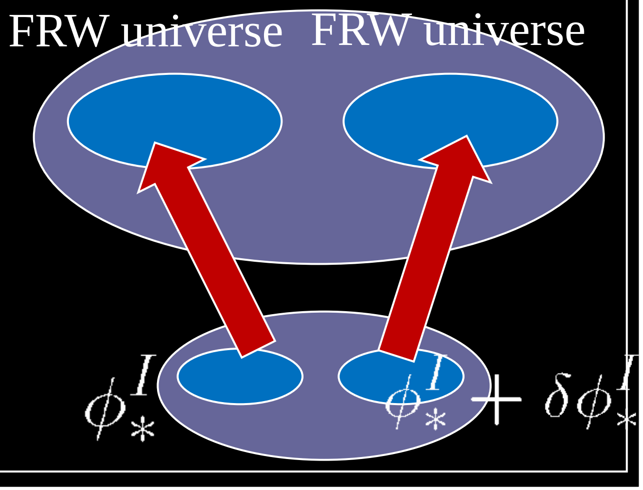
$\zeta = 0$ Flat slicing

Space

$$\boxed{\zeta(t,\mathbf{x}) = \delta N \equiv N(t,\mathbf{x}) - N_0(t)}$$

- Local expansion
= Expansion of the **unperturbed** Universe

$$N(t_c,\vec{x}) = N(t_c,\phi^I(t_c,\vec{x}))$$



FRW universe FRW universe

ϕ^I_* $\phi^I_* + \delta\phi^I_*$

◆ δ N formalism

[Ex] Single slow-roll scalar: $\dot{\phi} \approx -\frac{V'(\phi)}{3H}, \quad H^2 \approx \frac{V(\phi)}{3}$

***e*-folding number**

$$N = \int H dt = \int \frac{d\phi}{\dot{\phi}} H \quad \Rightarrow \quad \delta N = N_{,\phi} \delta\phi = \frac{H}{\dot{\phi}} \delta\phi$$

◆Calculation of Power spectrum

well known result

$$< \zeta\zeta > = N_{,\phi}^2 < \delta\phi\delta\phi > = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 = \left(\frac{H^2}{2\pi\dot{\phi}}\right)^2$$

- ➢Also *Easy* to calculate **Non-Gaussianity** by only using **background equation**
- ➢ **Applied to Multi-scalar system**

[①Violation of **single**]

◆ *Temporary violating of slow-roll condition*
[②**Violation of slow-roll**]

For Single inflaton-field (this talk)

● **Multi-field** inflation always shows (With Naruko, Sasaki in progress)

◆ ***δ N formalism*** $O(\epsilon^0)$ $\zeta(t,\mathbf{x}) = \mathrm{const}$

➢ Ignore the **decaying mode** of curvature perturbation

Not conserved !

◆ ***Beyond δ N formalism*** $O(\epsilon^2)$ $\zeta(t,\mathbf{x}) \cancel{=} \mathrm{const}$

➢ **Decaying modes** cannot be neglected in this case

➢ **Enhancement** of curvature perturbation in the **linear theory**

[Seto et al (01), Leach et al (01)]

◆ *Linear theory for a single scalar field*

● Mukhanov-Sasaki equation：Master equation

$$\mathcal{R}_c^{\mathrm{Lin}''} + 2\frac{z'}{z}\mathcal{R}_c^{\mathrm{Lin}'} + k^2 c_s^2 \mathcal{R}_c^{\mathrm{Lin}} = 0$$

☐ If we focus on a scalar type perturbation, One can derive the equation for One basic variable as one degree of freedom（Density）

Solution in Long wavelength expansion (gradient expansion)

$\boxed{O(k^0)}$ $\mathcal{R} = \mathrm{const}$ $\boxed{O(k^2)}$ $\boxed{\propto k^2 \int \frac{d\eta}{z^2}}$

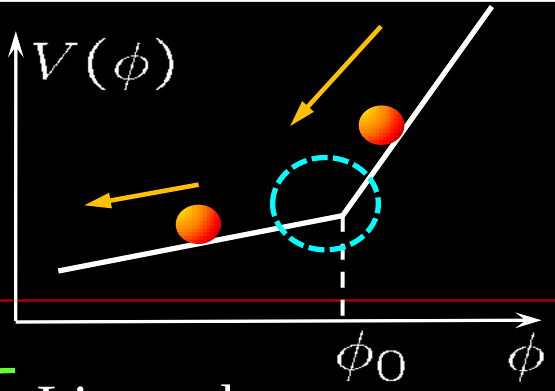
Growing mode

Decaying mode

◆ Example

- Starobinsky's model (92):

➢ **There is a stage at which slow-roll conditions are violated**



- Leach, Sasaki, Wands & Liddle (01)
  - Linear theory
  - The $\boxed{O(\epsilon^2)}$ in the expansion

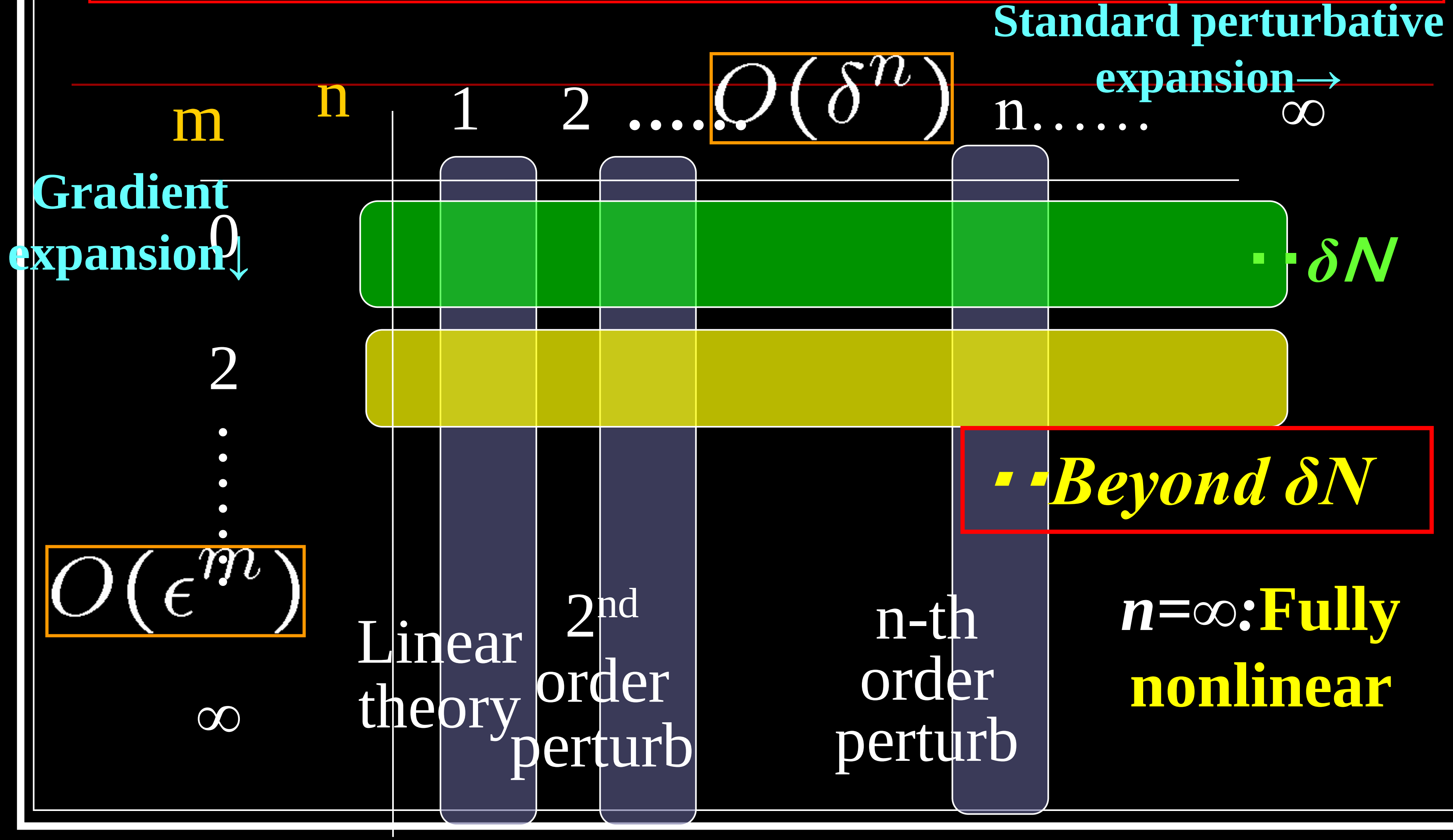
Violating of Slow-Roll
Decaying mode ⇒ $\mathcal{R}'(\eta) \neq 0$

⇒ $$\boxed{\mathcal{R}(0) = \alpha^{\mathrm{Lin}}\mathcal{R}(\eta_*)}$$

$\eta = 0$ (Final) η_* (initial)

◆ *Enhancement of curvature perturbation near* $\boxed{\phi = \phi_0}$

● *Nonlinear perturbations on superhorizon scales up to Next-leading order in the expansion*



Standard perturbative expansion→

m

n

1 2 …… $O(\delta^n)$ n…… ∞

Gradient expansion↓

0

2

⋮

$O(\epsilon^m)$

∞

⋅⋅ δN

⋅⋅Beyond δN

Linear theory

2nd order perturb

n-th order perturb

***n*=∞: Fully nonlinear**

◆ Beyond δN-formalism for single scalar-field

YT, S.Mukohyama, M.Sasaki
& Y.Tanaka JCAP(2010)

Simple result !

● Nonlinear theory in $\boxed{O(\epsilon^2)}$

- **Nonlinear variable** (including δN)
- **Nonlinear Source term**

$$\boxed{\mathcal{R}_c^{\mathsf{NL}\,\prime\prime} + 2\frac{z'}{z}\mathcal{R}_c^{\mathsf{NL}\,\prime} + \frac{c_s^2}{4}K^{(2)}[\,\mathcal{R}_c^{\mathsf{NL}}\,] = O(\epsilon^4)}$$

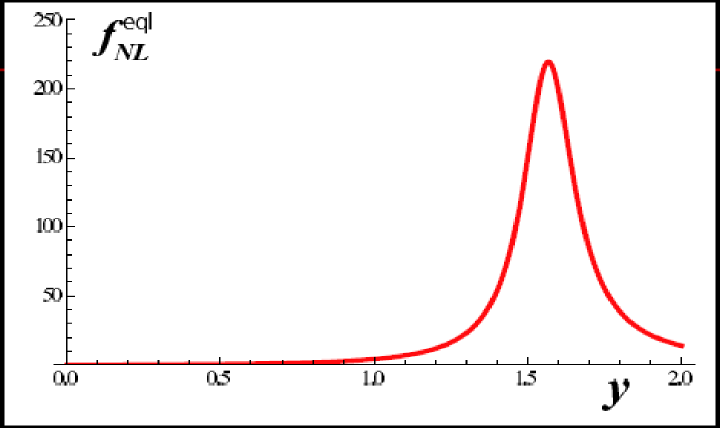
Ricci scalar of spatial metric

● Linear theory

$$\boxed{\mathcal{R}_c^{\mathsf{Lin}\,\prime\prime} + 2\frac{z'}{z}\mathcal{R}_c^{\mathsf{Lin}\,\prime} + k^2 c_s^2 \mathcal{R}_c^{\mathsf{Lin}} = 0}$$

◆ Application of *Beyond δN-formalism*

- To **Bispectrum** for **Starobinsky model**



$\boxed{f_{NL}^{eql} \simeq 2T}$: **Ratio of the slope of the potential**

$y = \sqrt{T}k/k_0 \simeq 1.5$

Even for $T = 10$

$f_{NL} \sim 20$ at $k \simeq 0.5k_0$

- *Temporary violation of slow-rolling* leads to particular behavior (*e.g. sharp spikes*) of the power & bispectrum

→ Localized feature models

☐ *These features may be one of powerful tool to discriminate many inflationary models with the future precision observations*

◆ *Beyond δN-formalism*

Single scalar case

System : $I = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G_N} + P(X, \phi) \right]$, $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

- *Single scalar field with general potential & kinetic term*

including K- inflation & DBI etc

● ADM decomposition & Gradient expansion

Small parameter: $\epsilon = 1/(HL)$ $\partial_i\psi = \psi \times O(\epsilon)$

◆ **Background is the flat FLRW universe**

Basic assumption:

$$\beta^i = O(\epsilon), \quad v^i = O(\epsilon), \quad \partial_t\tilde{\gamma}_{ij} = O(\epsilon) \Rightarrow \partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$$

- Absence of any decaying at leading order
- Can be justified in taking the background as FLRW

Solve the Einstein equation after ADM decomposition

● **General solution**

in Uniform Hubble + Time-orthogonal slicing

valid up to $\boxed{O(\epsilon^2)}$

YT& Mukohyama, JCAP 01(2009)

● ***Curvature perturbation***

Spatial metric $\gamma_{ij} = a^2 \boxed{e^{2\zeta}} \tilde{\gamma}_{ij}$; $\det(\tilde{\gamma}_{ij}) = 1$

$$\zeta \simeq \underset{\uparrow}{(\mathrm{const})} + \underbrace{(\mathrm{time}-\mathrm{dep})}_{\boxed{O(\epsilon^2)}} + O(\epsilon^4),$$

Constant (δ N)

$\boxed{O(\epsilon^2)}$

$\boxed{\delta P = c_s^2 \delta\rho + \rho\Gamma\delta\phi,}$

- **Variation of Pressure (speed of sound etc)**
- **Scalar decaying mode**

◆ Beyond δN-formalism for multi-scalar

With Naruko in progress

~~System : $I = \int d^4x\sqrt{-g}P(X^{IJ}, \phi^K), \quad X^{IJ} = -g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J$~~

- **ADM decomposition & Gradient expansion**

in Uniform Hubble + Time-orthogonal slicing

$$\boxed{\frac{\partial_t\psi}{\psi} = \frac{H}{2}(\alpha - 1)}$$

Curvature pertb. can be enhanced by inhomogeneous lapse

Using background solution with proper time $\tau \equiv \int_{x^i=const.} dt \;\; \alpha$

$$\boxed{\phi^g(t, x^i)\Big|_{\text{gradient}} = \phi_0(\tau)}$$

⇒ $$\boxed{\alpha^{(0)} = -\frac{2H(t)}{E^{(0)}(t) + P^{(0)}(t, x^i)}}$$

- Need to solve in a numerical way

$$\boxed{\alpha = f\left[\phi(\tau)\right] = f\left[\phi\left(\int dt\alpha\right)\right]}$$

- Analytic example (Naruko brid)

$$P = \frac{1}{2}X_{IJ} - V(\phi^I), \quad V(\phi^I) = V_0 \exp\left[\sum_I \frac{1}{2}m_I^2\phi_I^2\right].$$

In Uniform e-folding slicing

$$\mathcal{N} \equiv \frac{1}{3}\int_{x^i=const.} d\tau K = N(t) \quad N = -\int dt H(t)$$

Easy to obtain the solution because $\partial_t\psi(t, x^i) = 0$

◆ Gauge transformation to uniform Hubble can give an analytic solution of time-dependent curvature perturb.

◆ *Summary*

- We develop a theory of nonlinear cosmological perturbations on superhorizon scales for a scalar field with a general potential & kinetic terms
- We employ the **ADM formalism** and the spatial **gradient expansion** approach to obtain general solutions valid up through second-order $O(\epsilon^2)$
- We formulate a general method to match n-th order perturbative solution
- Can applied to Non-Gaussianity *in* ***temporary violating of slow-rolling***
- **Beyond δN-formalism:** *Two nonlinear effects*
- ① **Nonlinear variable : including δN (fully nonlinear)**
- ② **Nonlinear source term : Simple 2nd order diff equation**
- **Applications:** **Bispectrum** for **Starobinsky** model & **Inflaton stopping**