Chap.1 Kinematics and Dynamics of Galactic Stars

- Orbits of stars
  - Orbits in gravitational potentials, Integral of motion
- Kinematics of stars
  - Velocity components, LSR, solar motion, Galactic constants
- Distribution functions of stars
  - Schwarzschild, modeling distribution functions
- Jeans equations
  - Jeans theorem, spherical system, asymmetric drift
- Virial theorem

1. Orbits of stars

Galaxy structure: superposition of stellar orbits

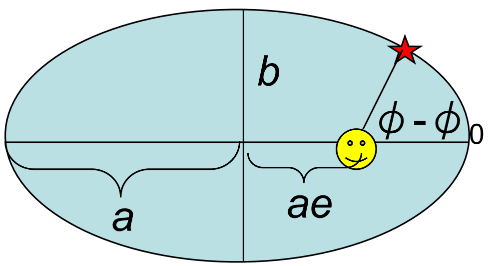
- Orbits in a spherical potential Φ(r)

  L*=*r × ***dr*/*dt*** =const.

  ⇒ confined to the 2D orbital plane ⇒ coordinates (r, ϕ)

E.g. Kepler potential (by a point mass: M) $\Phi(r) = -GM/r$

$$\begin{cases} L = r^2 d\phi/dt \\ E = \dfrac{1}{2}\left(\dfrac{dr}{dt}\right)^2 + \dfrac{L^2}{2r^2} - \dfrac{GM}{r} \end{cases}$$



⇒ $$\frac{a(1-e^2)}{r} = 1 + e\cos(\phi - \phi_0)$$ equation for an ellipse (orbit)

$e \equiv \sqrt{1 - 2|E|L^2/(G^2M^2)}$ eccentricity

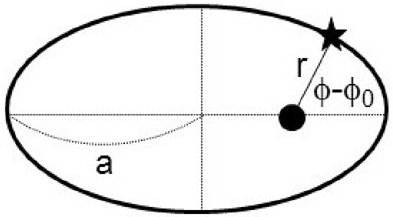
$a \equiv L^2/GM(1-e^2)$ semi-major axis

$r_{pr} = a(1-e)$ pericenter

$r_{ap} = a(1+e)$ apocenter

⇒ $$e = \frac{r_{ap} - r_{pr}}{r_{ap} + r_{pr}}$$

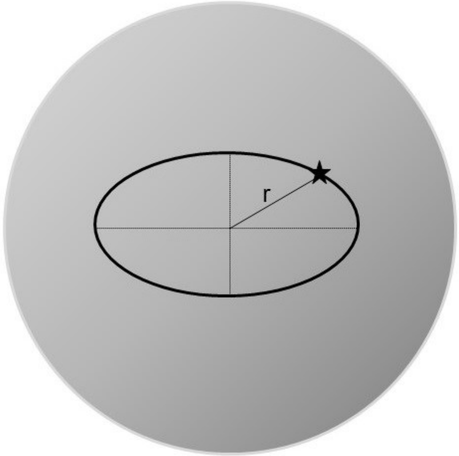
Kepler motion
(orbit in a point-mass potential)



$\Delta\phi = 2\pi$ over one period of radial oscillation

⇒ closed orbit

Orbit inside a uniform sphere



$M(<r) = 4\pi\rho r^3/3$

$F(r) = -GM(<r)/r^2$

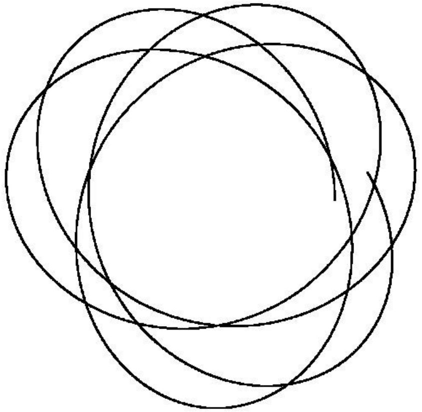
$\propto -r$

$F_x \propto -x$, $F_y \propto -y$

Simple oscillator

$\Delta\phi = \pi$

closed orbit



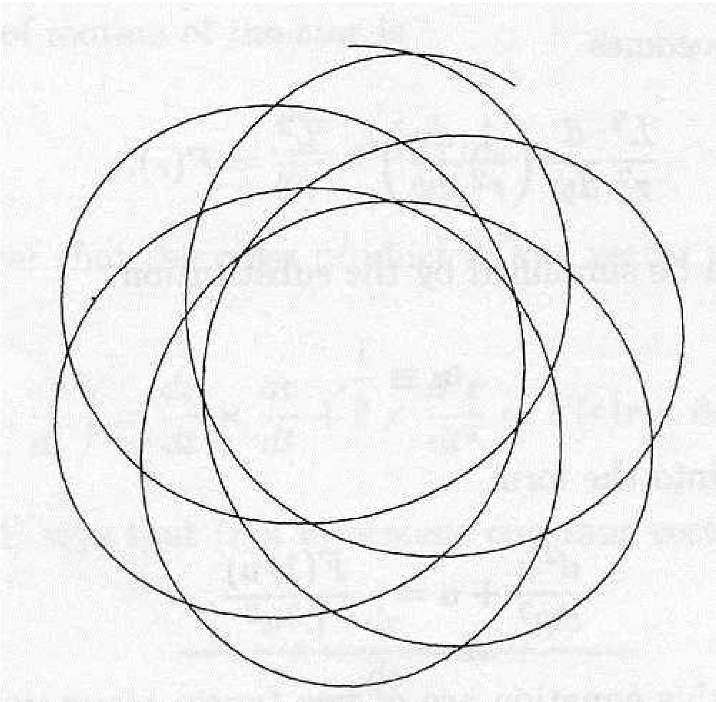
Orbit in a gravitational potential provided by a general spherical mass distribution

$\pi < \Delta\phi < 2\pi$

Rosette orbit

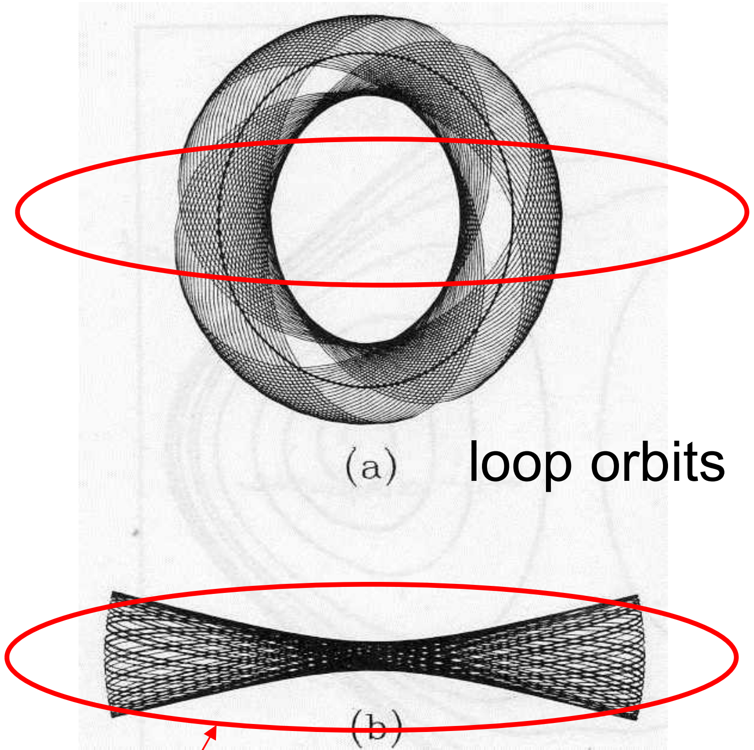
(non-closed)

· 2D axisymmetric $\Phi(R)$



loop orbits

· 2D non-axisymmetric $\Phi(x,y)$
(Nonrotating bar potential)



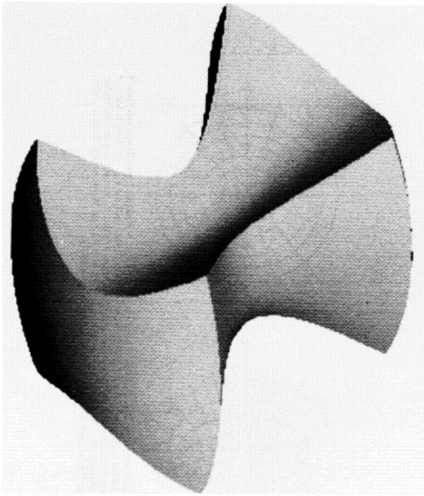
loop orbits

box orbits

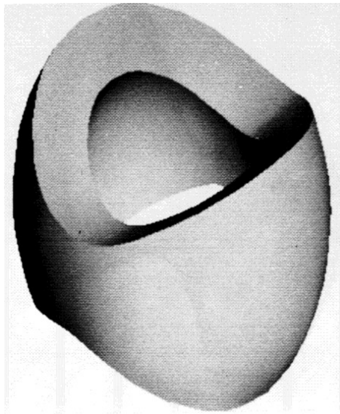
Orientation of a bar

Orbits in a (nonrotating) triaxial potential

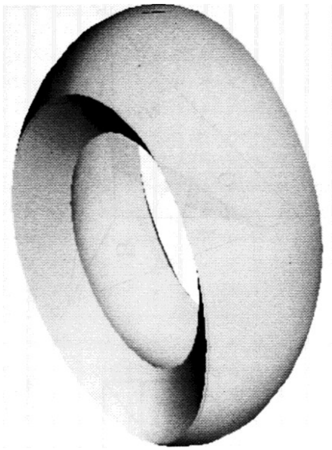
Box orbit



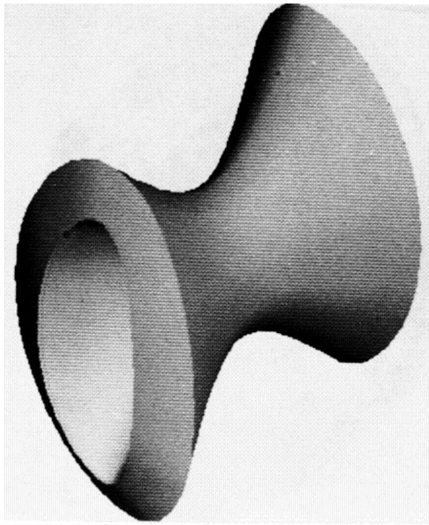
Short-axis tube orbit



Outer long-axis
tube orbit



Inner long-axis
tube orbit



Statler 1987, ApJ, 321, 113

- ## Stäckel potential
  - Hamilton-Jacobi eq. is separable ⇒ eq. of motions is solvable independently in each spatial coordinate
  - Integral of motion $I_i(\boldsymbol{x},\boldsymbol{v})$ i=1,3

  Only <u>regular orbits</u> exist (de Zeeuw 1985, MNRAS, 216, 273)

  explicit expressions for $I_1(\boldsymbol{x},\boldsymbol{v})$ (=E), $I_2(\boldsymbol{x},\boldsymbol{v})$, $I_3(\boldsymbol{x},\boldsymbol{v})$

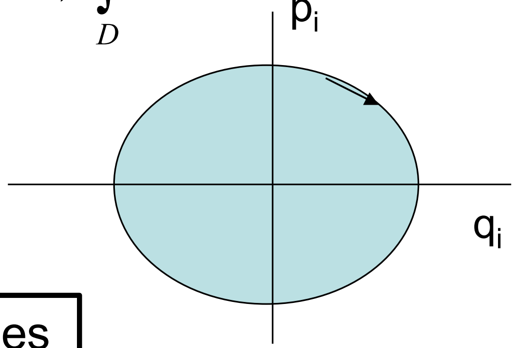
- ## Action integrals

$J_i(E,I_2,I_3)$ i=1,3

$$J_i = \frac{1}{2\pi}\oint p_i dq_i$$

  - Adiabatic invariance
  - $J_i \geq 0$ for bound orbits
  - Phase volume

$$V = \int_D d^3x d^3v = \int_D d^3J d^3\theta = (2\pi)^3 \int_D d^3J$$



Canonical transformation $(\boldsymbol{q},\boldsymbol{p}) \rightarrow (\boldsymbol{\theta},\boldsymbol{J})$

$$\dot{\theta}_i = \frac{\partial H}{\partial J_i} = \omega_i, \dot{J}_i = -\frac{\partial H}{\partial \theta_i} = 0$$

Action integrals for the classification of orbital families

Action integrals for a Kepler motion

$$J_r = \frac{1}{\pi}\int_{r_{\min}}^{r_{\max}} p_r dr = \frac{\sqrt{2}}{\pi}\int_{r_{\min}}^{r_{\max}} \sqrt{E - \frac{L^2}{2r^2} + \frac{GM}{r}}dr = \frac{GM}{\sqrt{2|E|}} - L = L\left[\frac{1}{\sqrt{1-e^2}} - 1\right]$$

$$J_\theta = \frac{1}{\pi}\int_{\theta_{\min}}^{\theta_{\max}} \sqrt{L^2 - \frac{p_\phi{}^2}{\sin^2\theta}}d\theta = L - |J_\phi|$$

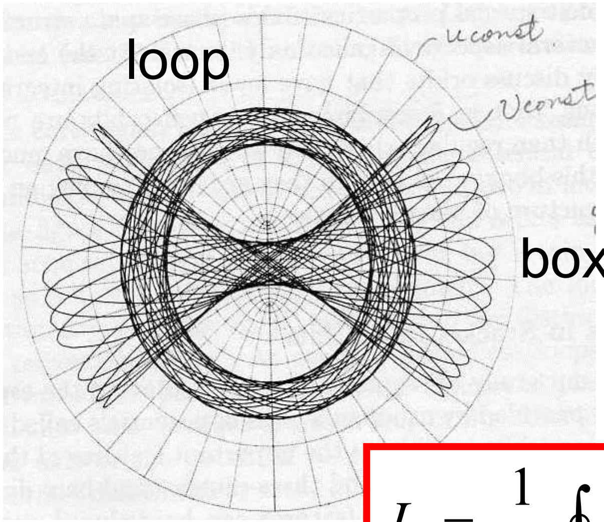
$$J_\phi = \frac{1}{2\pi}\oint p_\phi d\phi = p_\phi$$

$L=|J_\phi|+J_\theta$: **adiabatic invariance**

⇒ *Orbital eccentricity: e* is an adiabatic invariance as well

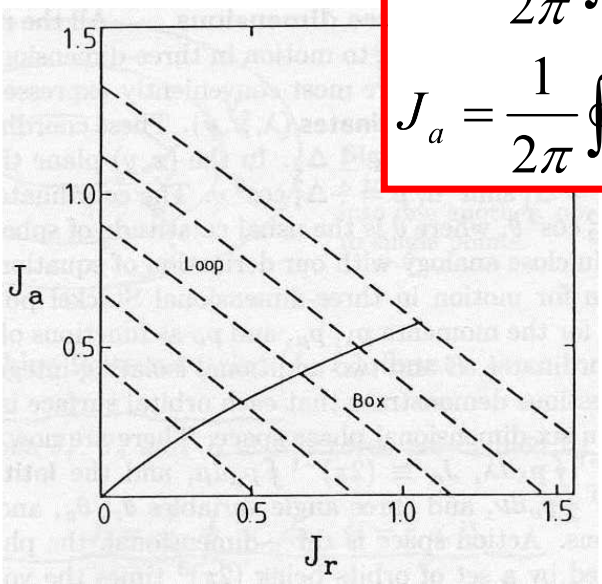
(a conserved quantity when the change of a gravitational potential is sufficiently slow compared to its dynamical time scale)

▪2D non-axisymmetric $\Phi(x,y) \Rightarrow \Phi(u,v)$

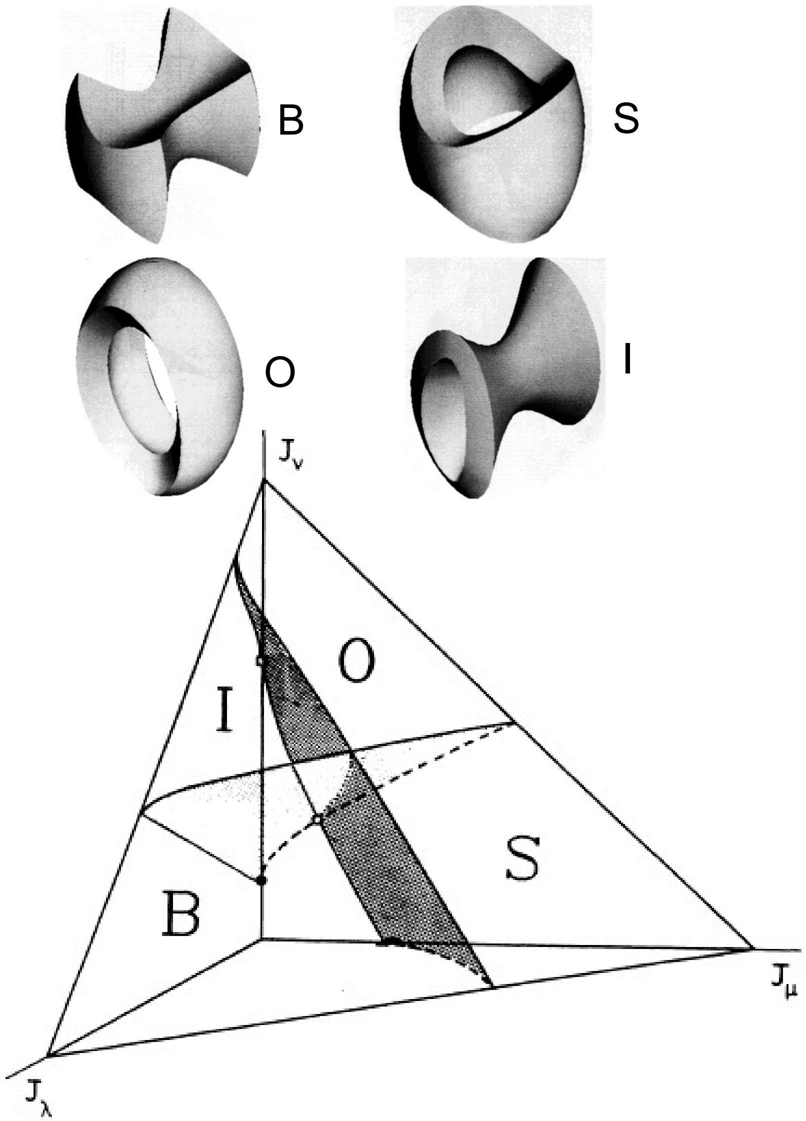


$$J_r = \frac{1}{2\pi} \oint p_u du$$

$$J_a = \frac{1}{2\pi} \oint p_v dv$$



▪3D triaxial $\Phi(\lambda,\mu,\nu)$

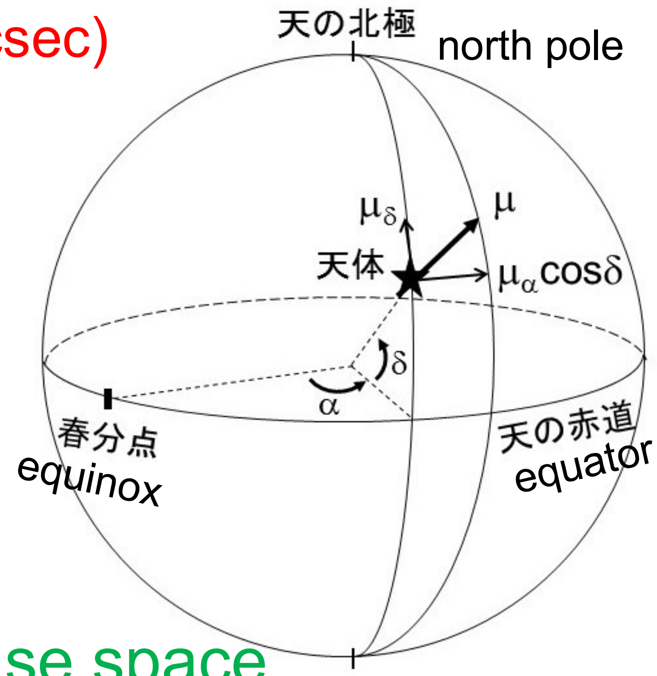


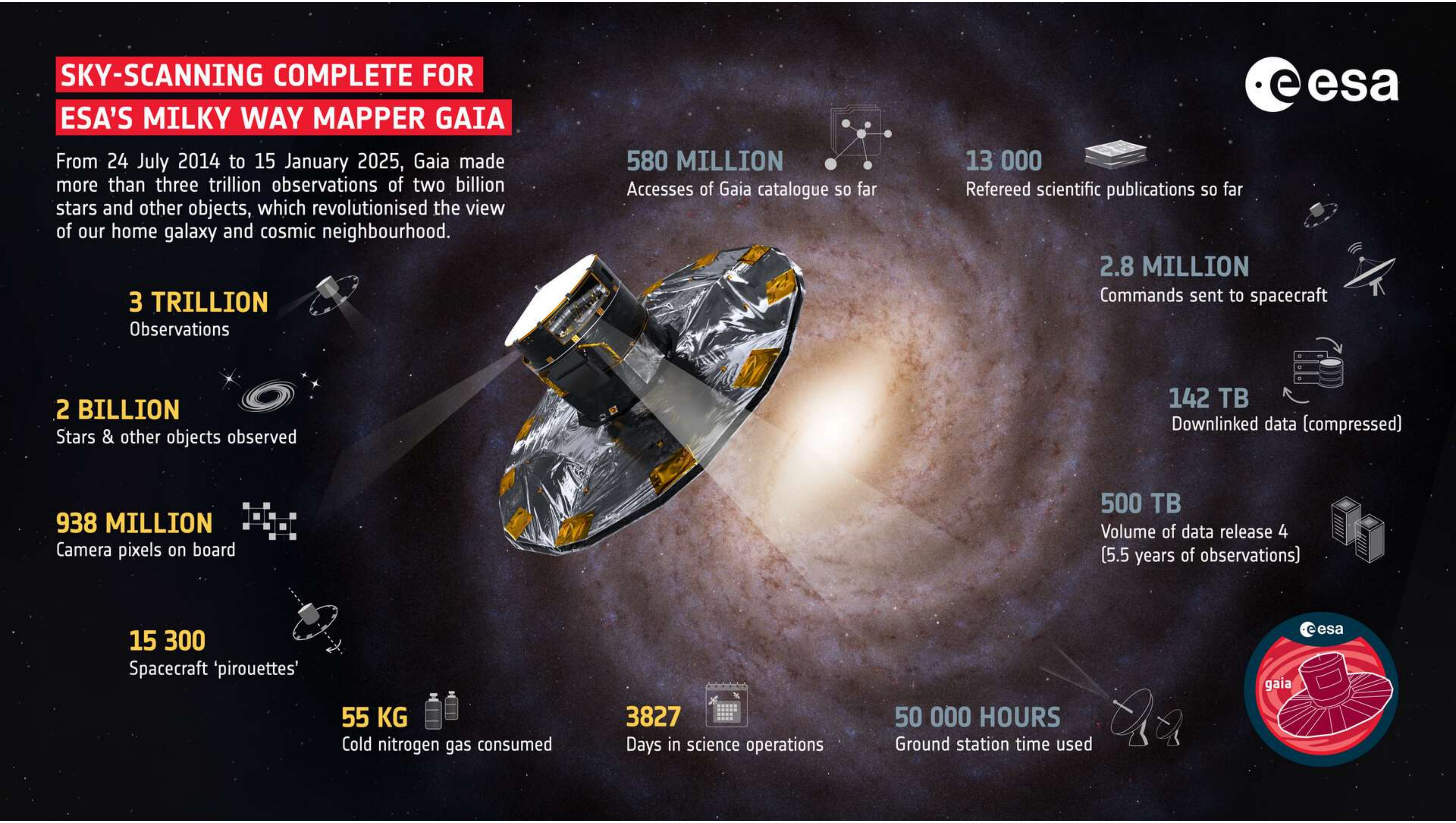
2. Kinematics of stars

Description of stellar kinematics observed from the Sun

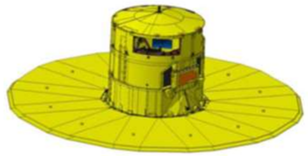
- Observed kinematical quantities
  - Line of sight velocity: V_{rad},
  - Distance: D (pc) = 1 / π (arcsec)
    or D (kpc) = 1 / π (mas, milli-arcsec)
  - Proper motion: $\mu = [(\mu_\alpha \cos\delta)^2 + (\mu_\delta)^2]^{1/2}$
    [unit: arcsec (″) /yr
    or mas (milli-arcsec: 10^{-3}″) /yr]

  Tangential velocity:
  V_{tan}(km/s)=4.74 D(pc) μ(arcsec/yr)
  = 4.74 D(kpc) μ(mas/yr)
  - (α,δ), D, V_{rad}, (μ_α,μ_δ)
  →3d position＋3d velocity→6d phase space




SKY-SCANNING COMPLETE FOR
ESA'S MILKY WAY MAPPER GAIA
From 24 July 2014 to 15 January 2025, Gaia made more than three trillion observations of two billion stars and other objects, which revolutionised the view of our home galaxy and cosmic neighbourhood.
esa
580 MILLION
Accesses of Gaia catalogue so far
13 000
Refereed scientific publications so far
3 TRILLION
Observations
2.8 MILLION
Commands sent to spacecraft
2 BILLION
Stars & other objects observed
142 TB
Downlinked data (compressed)
938 MILLION
Camera pixels on board
500 TB
Volume of data release 4
(5.5 years of observations)
15 300
Spacecraft 'pirouettes'
esa
gaia
55 KG
Cold nitrogen gas consumed
3827
Days in science operations
50 000 HOURS
Ground station time used

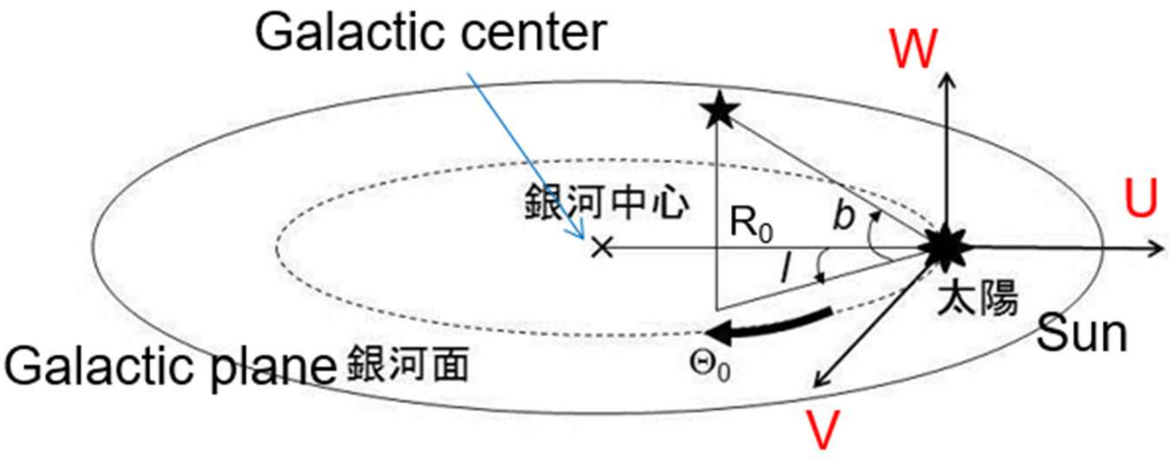
Astrometry Satellites: Hipparcos & Gaia



	Hipparcos	Gaia (DR3)
Operation period	1989-1993	2014-2025
Magnitude Limit	$V = 12.4$ mag	$G = 21$ mag
Number of stars	~120,000	~1.5 billion
Median astrometric accuracy	~ 1 mas ($H_p < 9$ mag)	20-30 μas ($G < 15$ mag)
		70 μas (at $G = 17$ mag)
		500 μas (at $G = 20$ mag)
		1.3 mas (at $G = 21$ mag)
Radial velocity accuracy	None	6.4 km s^{-1} (at $G_{RVS} = 14$ mag)
Astrophysical parameters (T_{eff}, log g, [M/H])	None	Available ($G < 17.6$ mag)

Gaia: 10μas = 10% error @distance 10kpc, 10μas/yr = 1km/s @20kpc
Hipparcos: 1mas = 10% error @distance 100pc, 1mas/yr = 5km/s @ 1kpc

The Local Standard of Rest (LSR) & UVW velocities



- The LSR: a circular orbit at ($R=R_0$, $V_{rot} = \Theta_0$)
- (U,V,W): star's velocities relative to the LSR
- (U_{sun},V_{sun},W_{sun}): solar motion relative to the LSR
- Observed star's velocities from the Sun (heliocentric)

  ($U-U_{sun}$, $V-V_{sun}$, $W-W_{sun}$) $\simeq$ (V_R, $V_\Phi - \Theta_0$, V_z) in cylindrical coords.

Galactic constants: Θ_0, R_0, (U_{sun},V_{sun},W_{sun})

Determination of Galactic Constants

LSR
(Θ_0, R_0)

- Rotational velocity of the LSR: Θ_0
  - Oort constants (A,B) → Ω_0=A-B → R_0 given → $\Theta_0=R_0$ (A-B)
  - Motion relative to Pop II system, but $\langle\Theta\rangle=0$ is assumed
  - Proper motion of Sgr A* → R_0 → Θ_0

    if Sgr A* is fixed at the center and the LSR has Θ_0=220km/s, then $\Theta_0=4.74D\mu_\ell$ → proper motion along Galactic long.: μ_ℓ ~ 5.8 mas/yr
- Solar position: R_0
  - The center of halo tracers (GCs, RR Lyr, Mira variables in the bulge)
  - Parallax of Sgr A*: p(mas) = $(D/\mathrm{kpc})^{-1}$ = 0.1 mas
  - Stellar motions near Sgr A* ("binary method") Salim & Gould 1999

➢ Kerr & Lynden-Bell (1986, MN, 221, 1023)

Θ_0=220 km/s, R_0=8.5 kpc (IAU standards)

➢ Recent trend: Θ_0 > 220 km/s, R_0 ~ 8 kpc

Solar motion
(U_{sun}, V_{sun}, W_{sun})

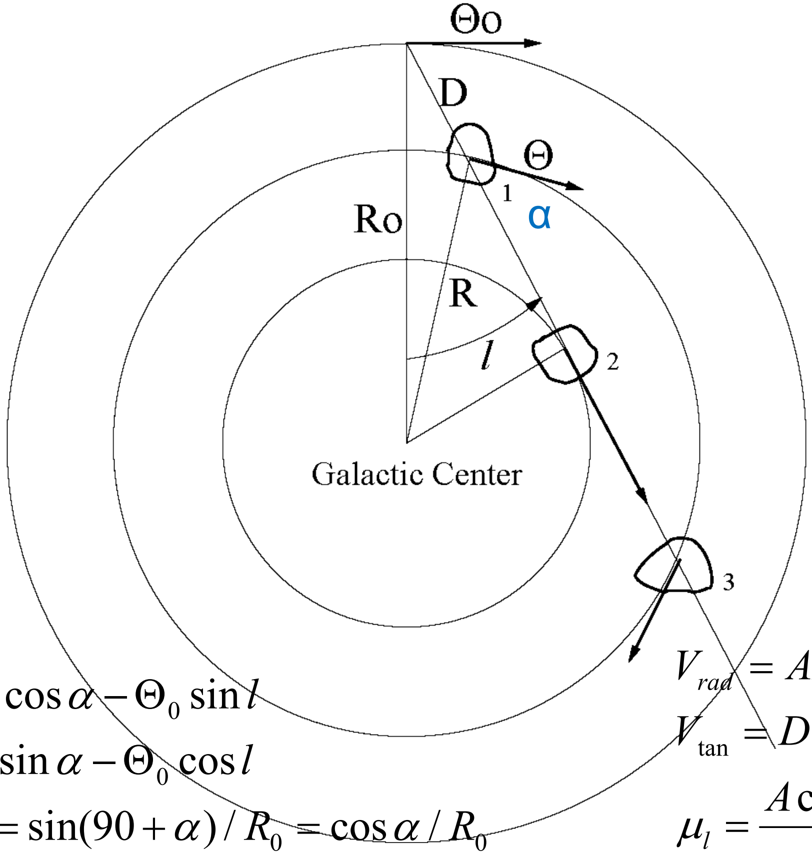
* Delhaye 1965 using A stars, K giants, M dwarfs

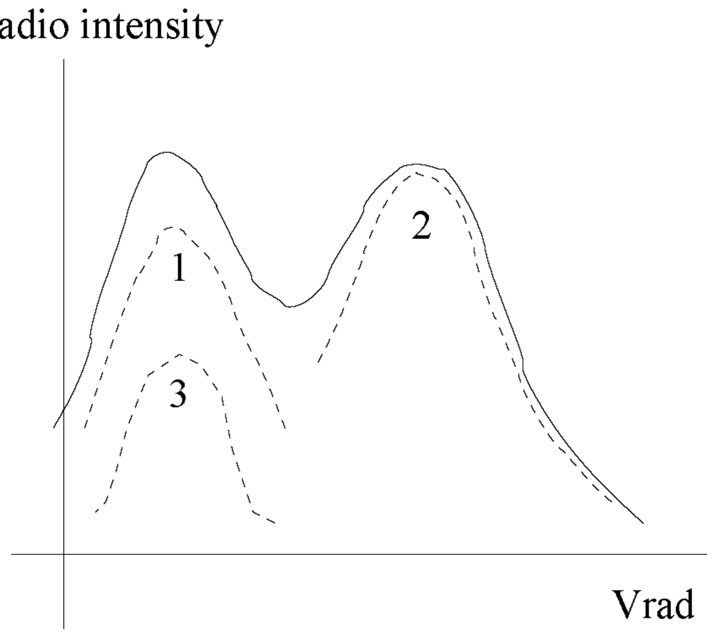
(U_{sun}, V_{sun}, W_{sun}) = (-9, 12, 7) km/s, (l,b)=(53,25)

* More recent result (Schönrich +10, Coşkunoğglu+11)

(U_{sun}, V_{sun}, W_{sun}) = (-11.10, <u>12.24</u>, 7.25) km/s

Determination of the rotation curve





$$V_{rad} = \Theta\cos\alpha - \Theta_0 \sin l$$
$$V_{\tan} = \Theta\sin\alpha - \Theta_0 \cos l$$
$$\sin l / R = \sin(90+\alpha)/R_0 = \cos\alpha / R_0$$
$$R\cos(90-\alpha) = R\sin\alpha = R_0\cos l - D$$
$$V_{rad} = (\Omega - \Omega_0) R_0 \sin l$$
$$V_{\tan} = (\Omega - \Omega_0) R_0 \cos l - \Omega D$$

$\Rightarrow$ $(R_0-R)/R_0 << 1 \Rightarrow$

$$V_{rad} = AD\sin 2l$$
$$V_{\tan} = D(A\cos 2l + B)$$
$$\mu_l = \frac{A\cos 2l + B}{4.74}$$
$$A \equiv \frac{1}{2}\left[\frac{\Theta_0}{R_0} - \left(\frac{d\Theta}{dR}\right)_{R_0}\right]$$
$$B \equiv -\frac{1}{2}\left[\frac{\Theta_0}{R_0} + \left(\frac{d\Theta}{dR}\right)_{R_0}\right]$$

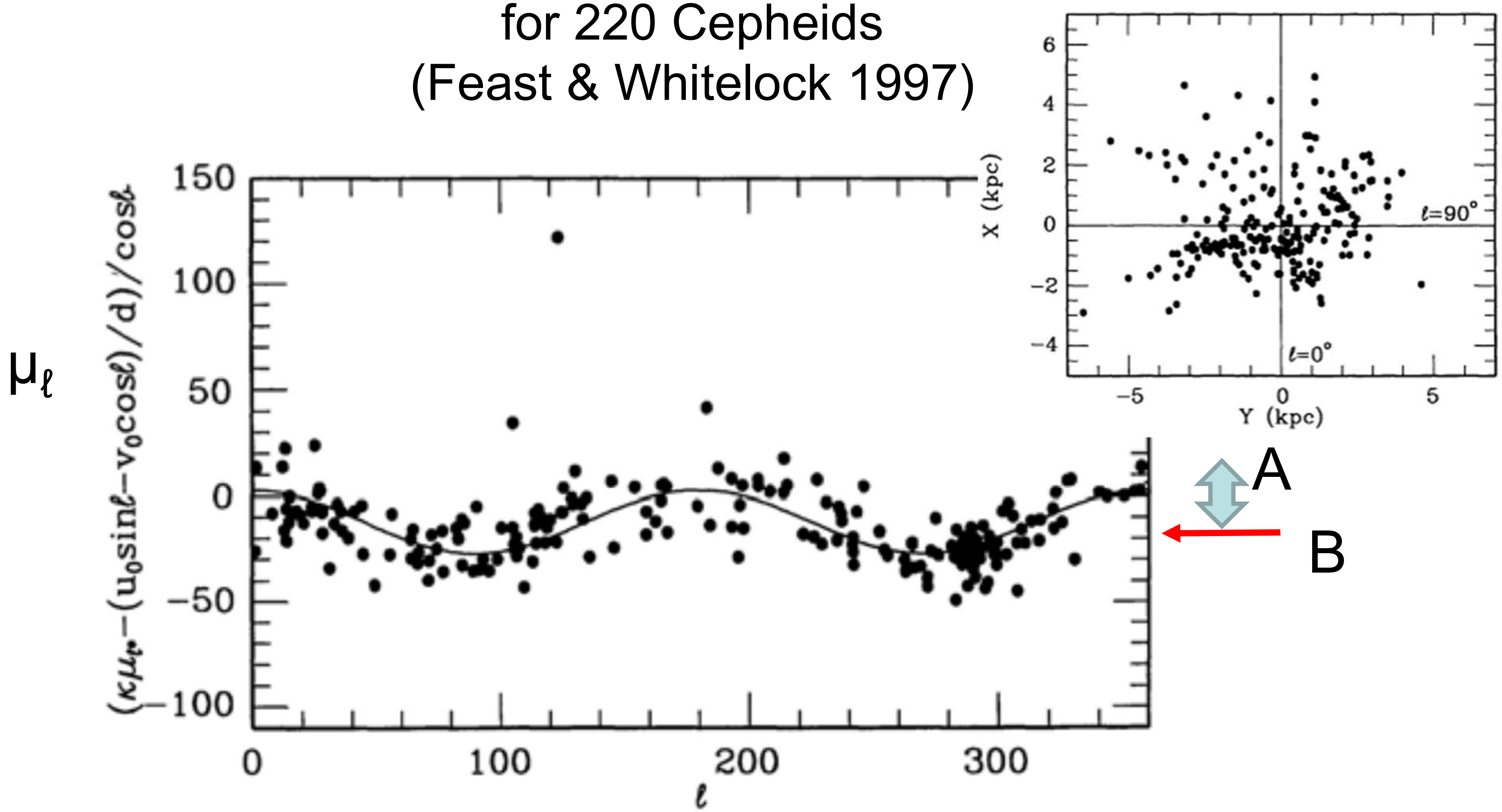
A, B: Oort Constants

$$\Omega_0 = \Theta_0 / R_0 = A - B$$
$$\left(\frac{d\Theta}{dR}\right)_{R_0} = -(A+B)$$

V_{rad} is max at $R = R_0 \sin l \Rightarrow \Theta(R)$

$$\frac{\partial V_{rad}}{\partial D} = 0 \Rightarrow \partial R/\partial D = 0$$
$$R^2 = D^2 + R_0^2 - 2DR_0\cos l$$
$$\Rightarrow D = R_0\cos l \quad \Rightarrow R = R_0 \sin l$$

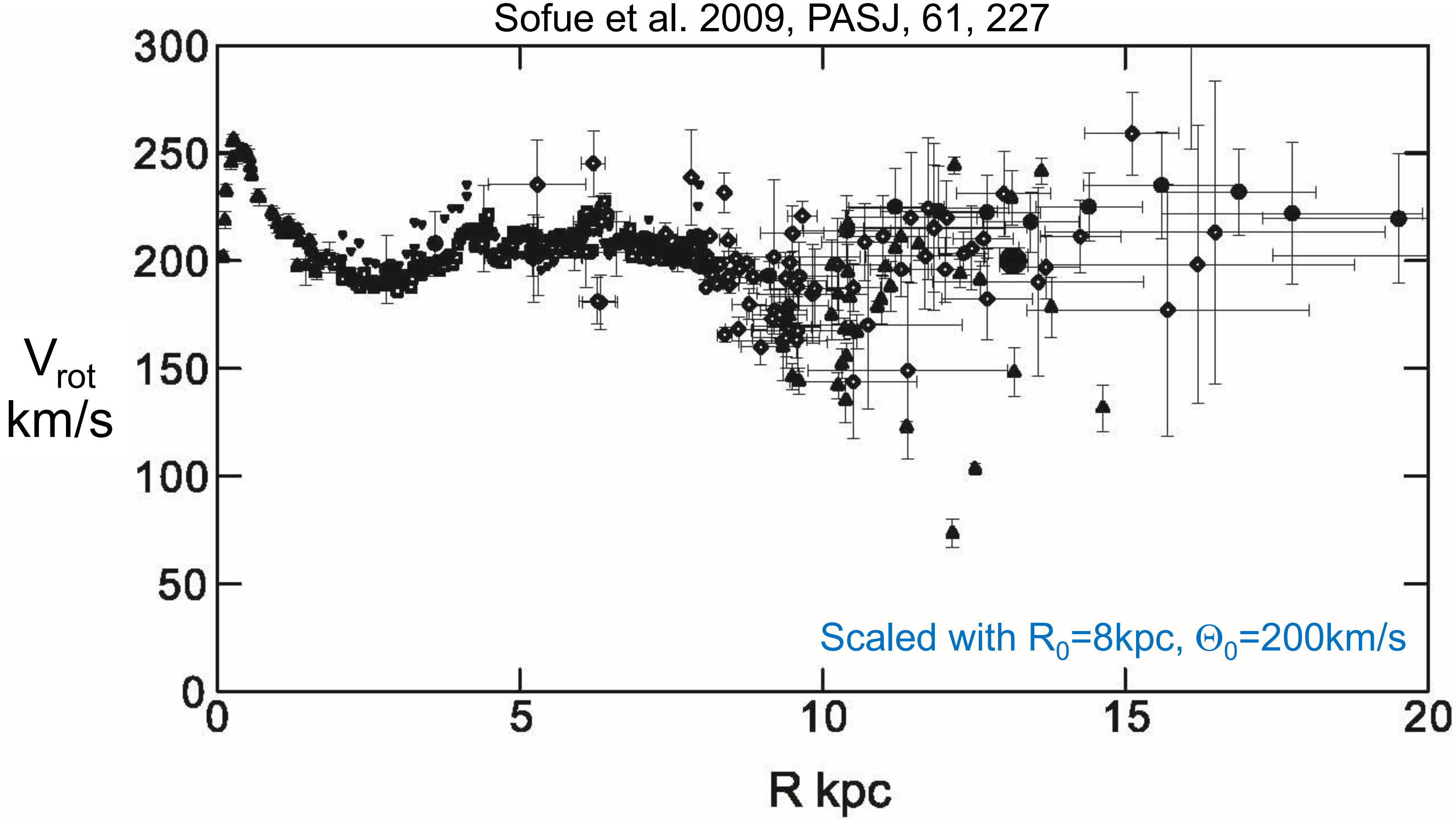
Hipparcos proper motions toward Galactic longitude
for 220 Cepheids
(Feast & Whitelock 1997)



$\mu_\ell \propto V_{tan}/D = A\cos 2\ell + B \Rightarrow A, B$

Rotation curve of the Milky Way

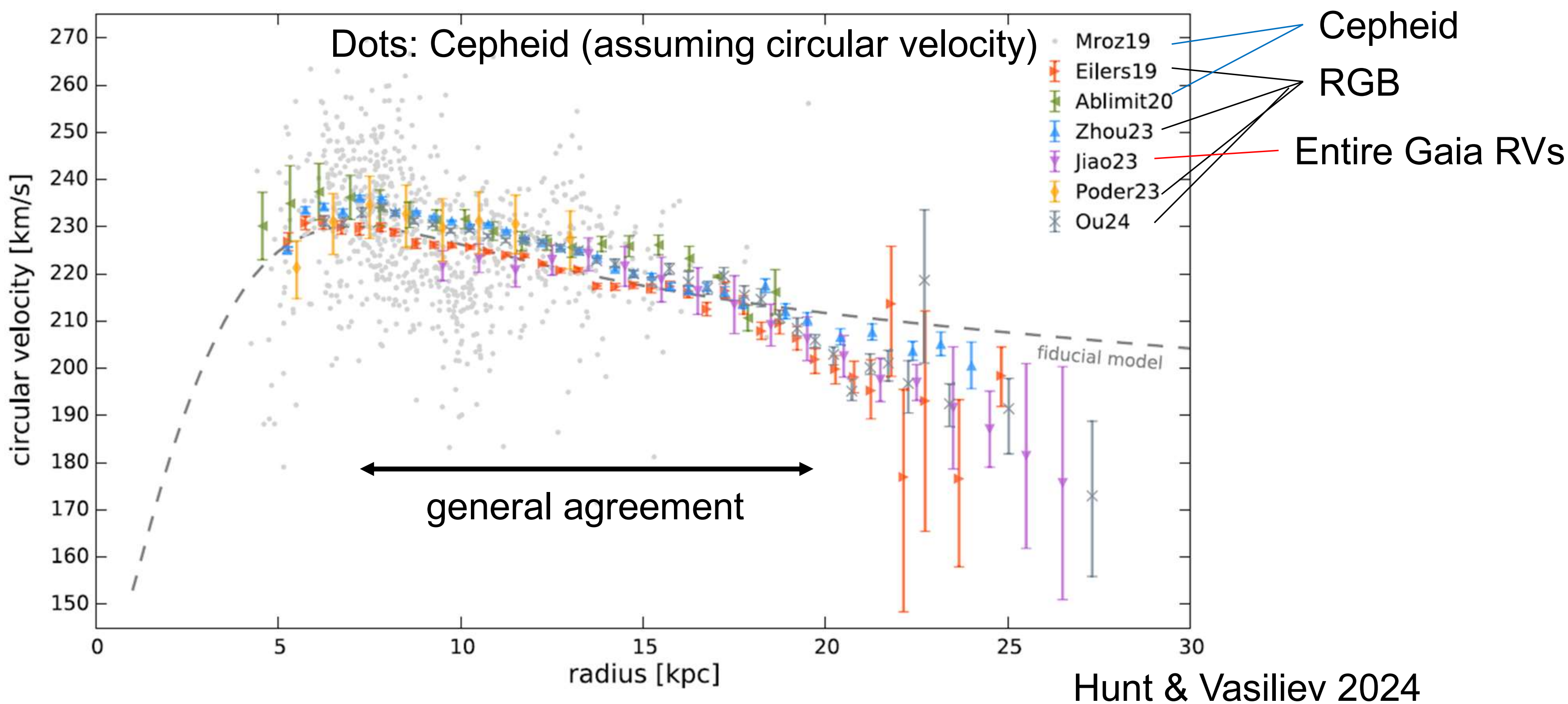
Sofue et al. 2009, PASJ, 61, 227



See also, Gunn, Knapp, Tremaine 1979, AJ, 84, 1181;
Fich & Tremaine 1991, ARAA, 29, 409

Rotation curve of the Milky Way

Recent results using Gaia data



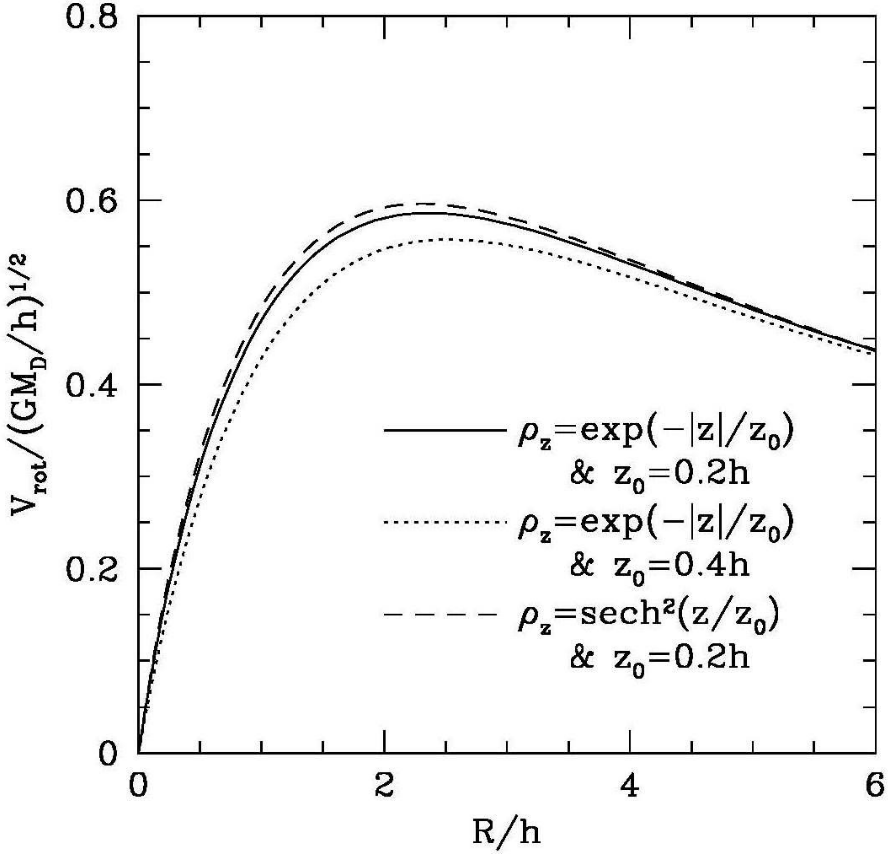
Hunt & Vasiliev 2024

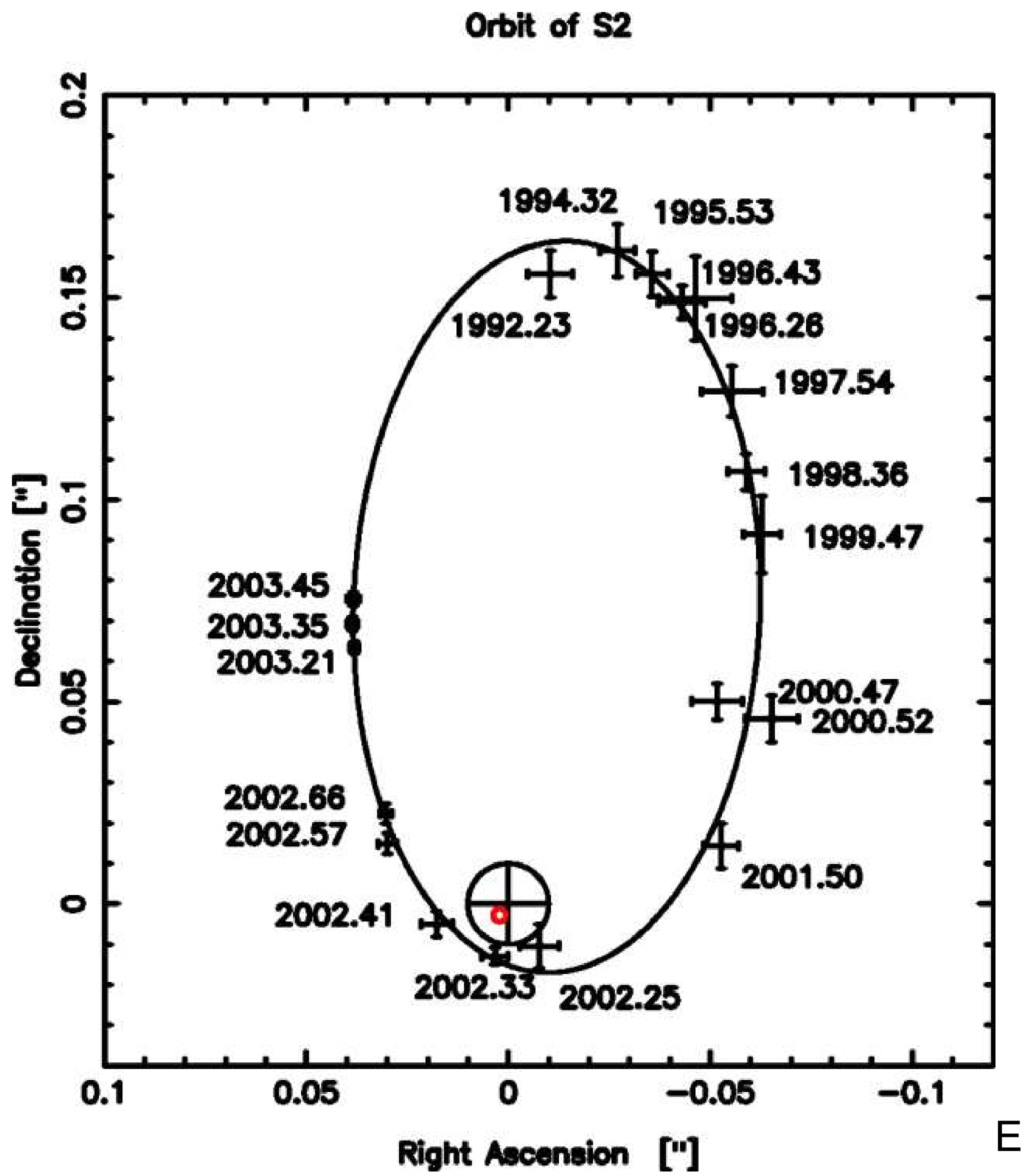
Fig. 17. A compilation of measurements of the Milky Way circular-velocity curve in the radial range 5–30 kpc. The dashed line shows the same fiducial model as in the previous plot.

But this declining rotation curve suggests only M_tot ~ a few times 10^{11} Msun,
that is inconsistent with other estimates of M_tot ~ 10^{12} Msun
⇒ circular velocity assumption?, dynamical equilibrium?

Effect of disk thickness on rotation curve at the disk plane

V_{rot} provided by an exponential disk: $\rho(R,z) = \rho_0 \exp(-R/h)\rho_z(z)$





Geometric determination of the distance to the Galactic Center

orbital eclipse vs. angular sep. → R_0

$R_0 = 7.94 \pm 0.42$kpc

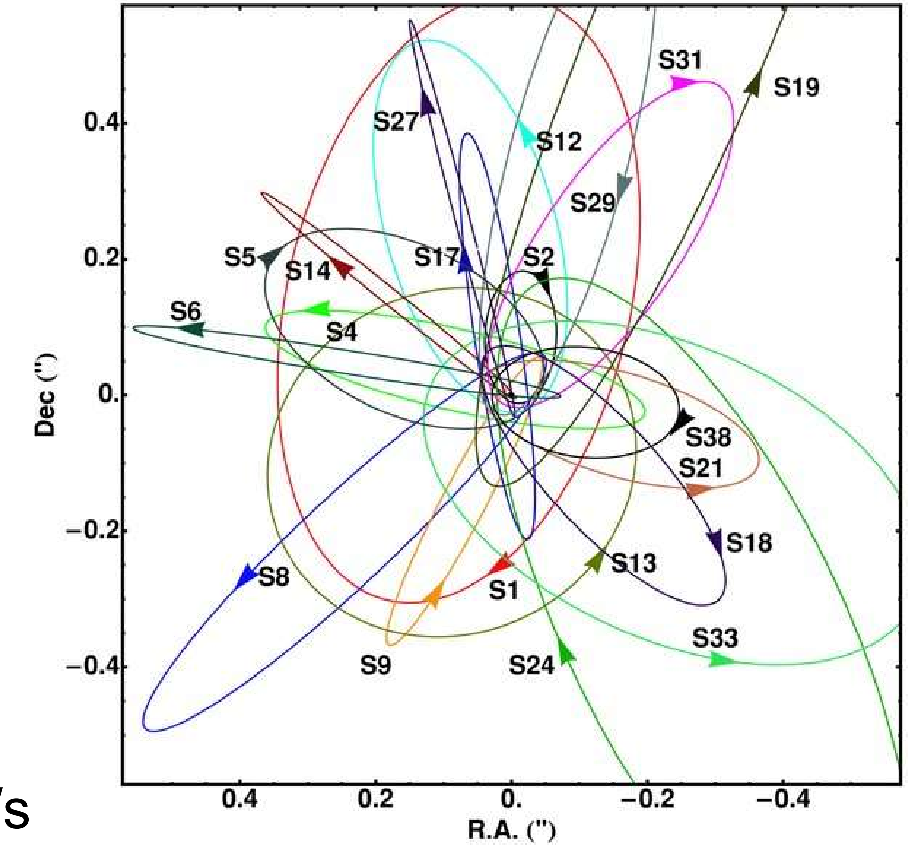
Eisenhauer et al. 2003, ApJ, 597, L121

Using more orbits

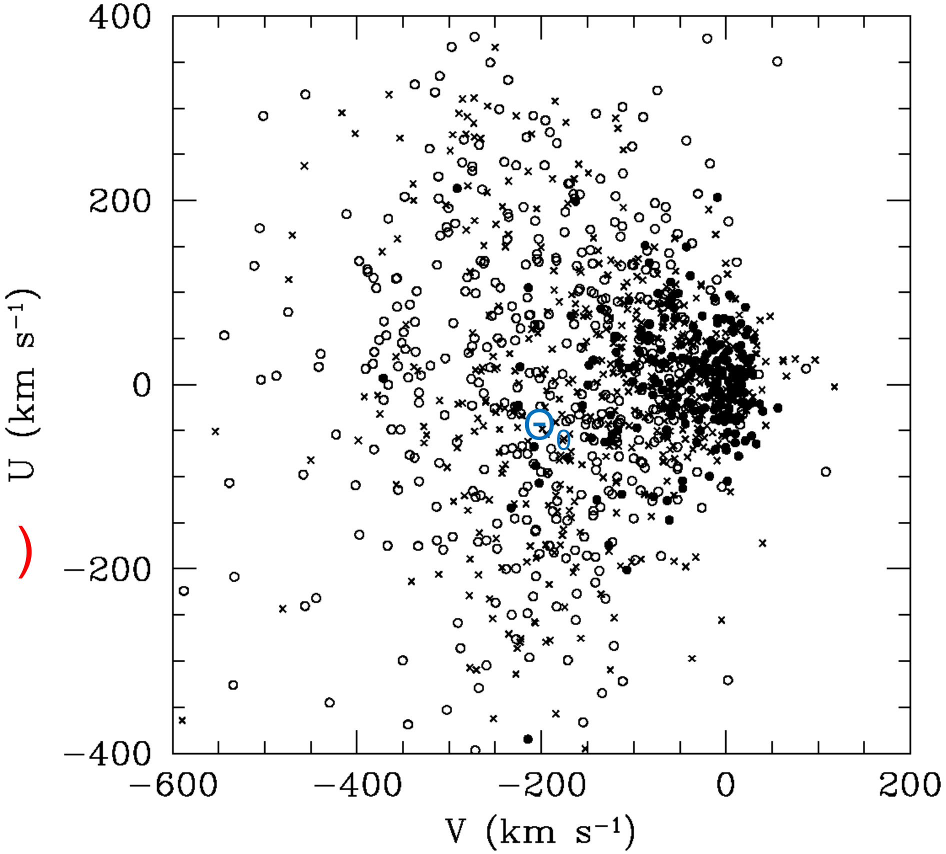
Gillessen et al. 2009:
16 years of monitoring
the orbits of 28 stars
$R_0 = 8.33 \pm 0.35$ kpc

Reid & Brunthaler 2004:
μ_ℓ(SgrA*)=6.379±0.026mas/yr

$\Rightarrow(\Theta_0+V_{sun})/R_0$=30.24 km/s/kpc
Then if R_0=8.3 kpc & V_{sun}=12.24 km/s
$\Rightarrow\Theta_0$=239 km/s



(U,V) velocities for nearby stars (V_R , $V_\phi - \Theta_0$)

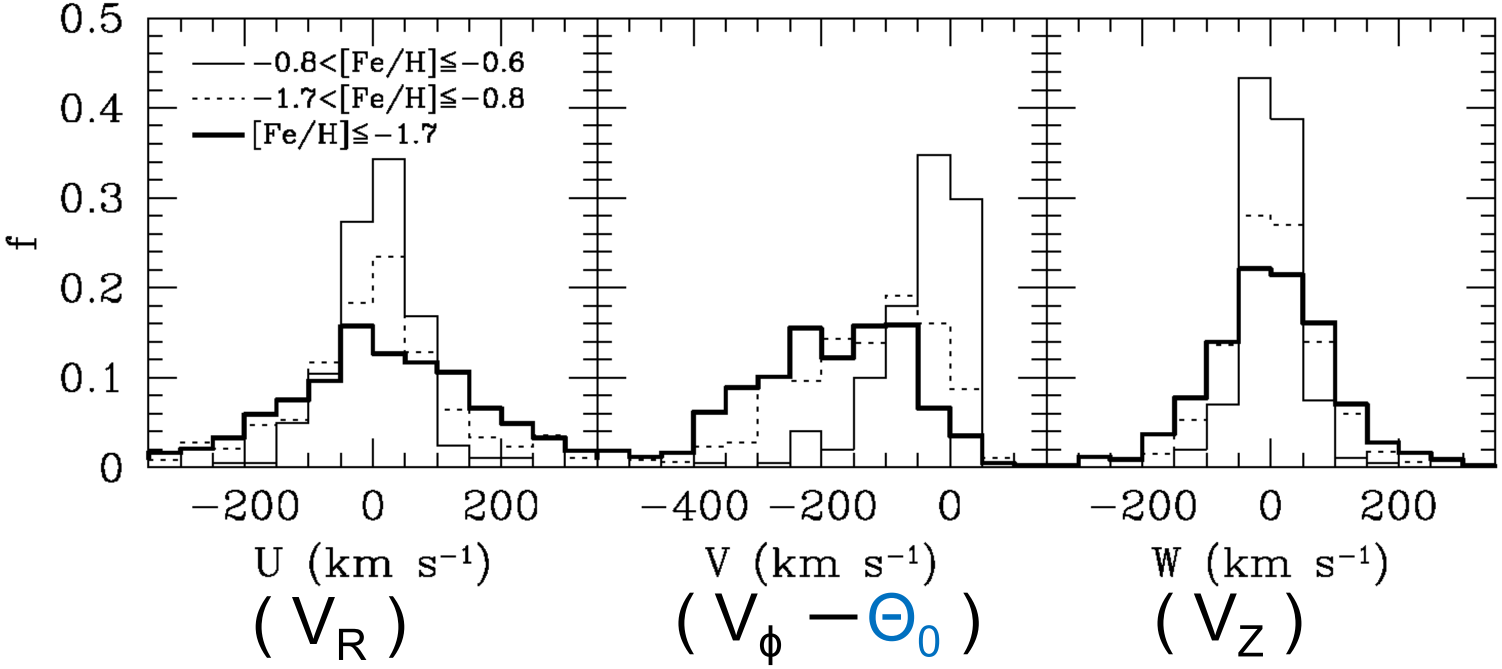


(V_R)

($V_\phi - \Theta_0$)

Distance from the Sun
D<4kpc

● -0.8<[Fe/H]≤-0.6
× -1.7<[Fe/H]≤-0.8
○ [Fe/H]≤-1.7



	σ_U	σ_V	σ_W	<V>
[Fe/H]≤-1.7	150 km/s	110 km/s	100 km/s	-200 km/s
-0.8<[Fe/H]≤-0.6	60 km/s	60 km/s	40 km/s	-30 km/s

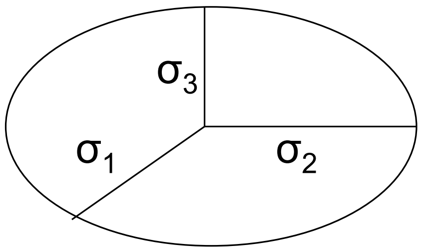
3. Distribution function of stars

- Schwarzschild (1907) model

$$f(v_1, v_2, v_3)dv_1dv_2dv_3 = \frac{dv_1dv_2dv_3}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3}\exp\left[-\left(\frac{v_1^2}{2\sigma_1^2}+\frac{v_2^2}{2\sigma_2^2}+\frac{v_3^2}{2\sigma_3^2}\right)\right]$$

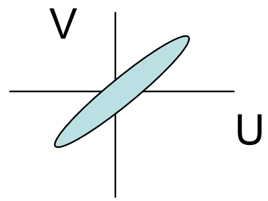
$\sigma_i^2 = \langle(v_i - \langle v_i\rangle)^2\rangle$ velocity dispersion

Velocity ellipsoid



Vertex deviation

σ_i axis does not necessary match the direction of (U,V,W)



Modeling distribution functions

$f(\boldsymbol{r}, \boldsymbol{v}, t) d^3\boldsymbol{r} d^3\boldsymbol{v}$ $(\boldsymbol{r}, \boldsymbol{v})$ phase space

$E(\boldsymbol{r}, \boldsymbol{v})$ $I_2(\boldsymbol{r}, \boldsymbol{v})$ $I_3(\boldsymbol{r}, \boldsymbol{v})$ Integrals of motions

$$n(\boldsymbol{r}) = \int f d^3\boldsymbol{v} \qquad \langle v_i \rangle = \frac{1}{n} \int v_i f d^3\boldsymbol{v}$$

$$\sigma_i^2 = \langle (v_i - \langle v_i \rangle)^2 \rangle \qquad \sigma_{ij}^2 = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$$

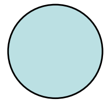
Integrals of motions, $I(\boldsymbol{r}, \boldsymbol{v})$, are the solutions to the steady-state collisionless Boltzmann equation

$$\frac{dI}{dt} = \boldsymbol{v} \cdot \nabla I - \nabla \Phi \cdot \frac{\partial I}{\partial \boldsymbol{v}} = 0$$

⇒ Jeans Theorem: $f(E, I_2, I_3)$, $f(J_1, J_2, J_3)$

Some simple cases

- f(E) isotropic velocity distribution

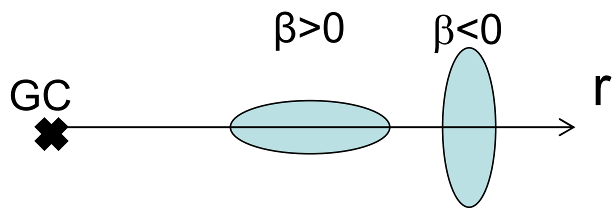


- f(E,L_z) $L_z = Rv_\phi$ in axisymmetric $\Phi(R,z)$
  - $\sigma_R^2 = \sigma_z^2$ (but $\neq \sigma_\phi^2$) anisotropic
  - but $\sigma_R^2 \neq \sigma_z^2$ near the Sun $\rightarrow$ presence of I_3

  $(\sigma_U, \sigma_V, \sigma_W) \approx (150,110,100)$ km/s for halo stars
- f(E,L) L: total angular momentum
  - $v_r = v\cos\eta$, $v_\theta = v\sin\eta\cos\psi$, $v_\phi = v\sin\eta\sin\psi$
  - $v_t^2 = v_\theta^2 + v_\phi^2 = v^2\sin^2\eta$, $L = |rv_t| = |rv\sin\eta|$
  - $\sigma_\theta^2 = \sigma_\phi^2 \neq \sigma_r^2$ anisotropic
  - $\beta(r) = 1 - \sigma_\theta^2/\sigma_r^2$, $\beta \leq 1$

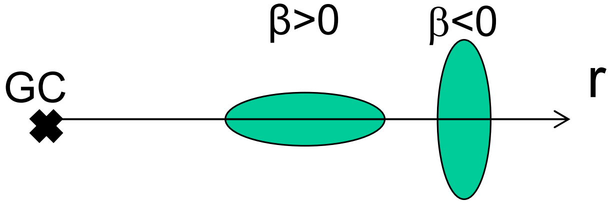
    $\beta > 0$: radially anisotropic

    $\beta < 0$: tangentially anisotropic

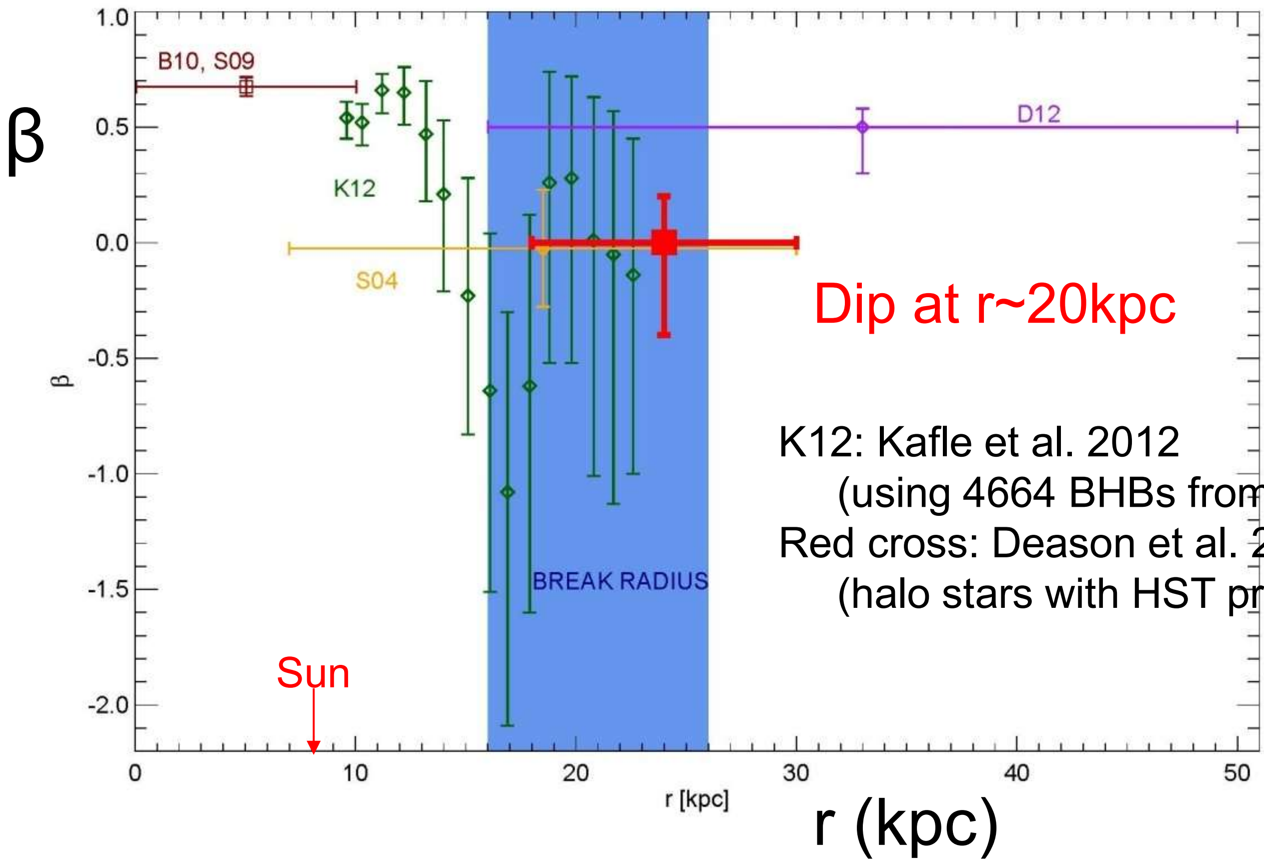


These velocity anisotropies reflect past merging/accretion histories

Velocity anisotropy parameter β(r)



Radial anisotropy:
$\beta(r)=1-(\sigma_\theta^2+\sigma_\phi^2)/2\sigma_r^2$



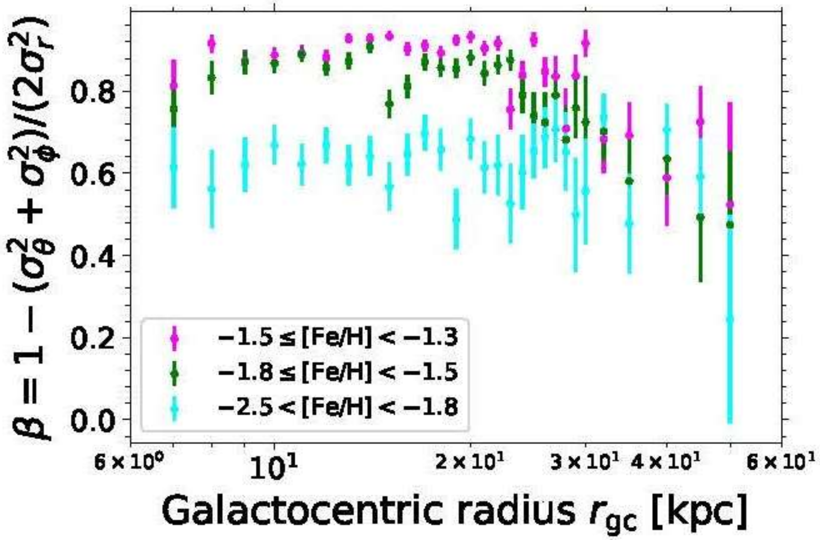
Dip at r~20kpc

K12: Kafle et al. 2012
(using 4664 BHBs from SDSS)
Red cross: Deason et al. 2013
(halo stars with HST proper motions)

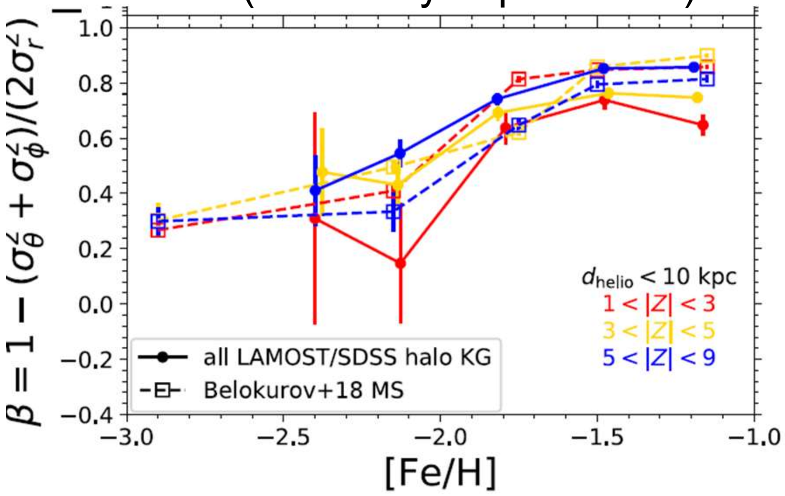
Recent results on β(r)

Bird et al. 2019
using 5600 K giants from
LAMOST and Gaia DR2

- -1.8<[Fe/H]<-1.3: boundary at r ~ 20kpc (effect of Sgr dwarf?)
- [Fe/H]<-1.8: no dip

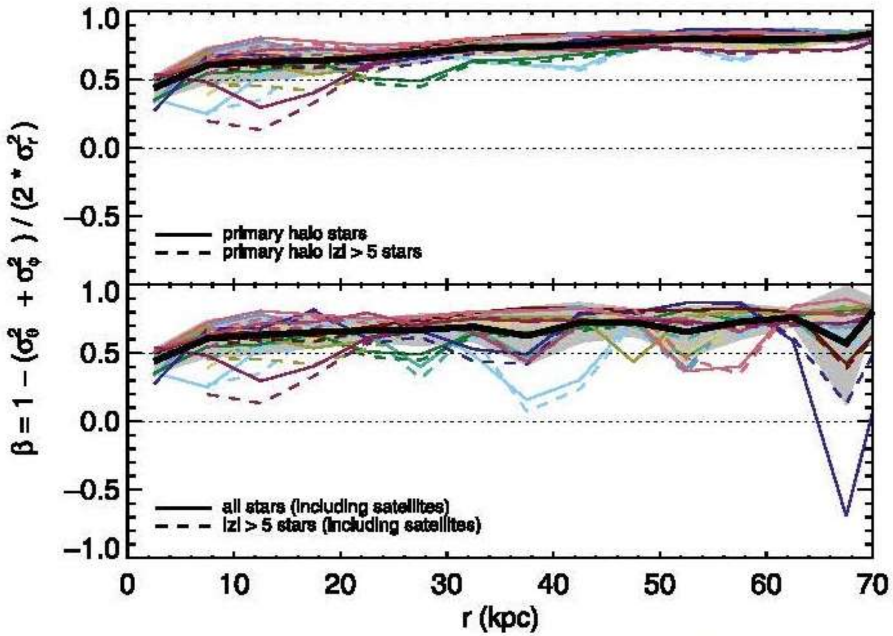


Bird et al. 2021
(metallicity dependence)

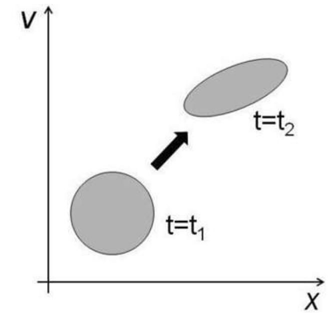


Loebman et al. 2019
using simulation results
by Bullock & Johnston 2005
(hierarchical clustering process)

β ~ 0.7
Radially anisotropic over entire radii
Presence of temporal dips



4. Jeans equations



Continuity eq. in phase space

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \frac{\partial (f \dot{x}_i)}{\partial x_i} + \sum_{i=1}^{3} \frac{\partial (f \dot{v}_i)}{\partial v_i} = 0$$

Boltzmann equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0$$

moment over v_i

$$\frac{\partial n}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (n \langle v_i \rangle) = 0$$

$$\frac{\partial n \langle v_j \rangle}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (n \langle v_i v_j \rangle) + n \frac{\partial \Phi}{\partial x_j} = 0$$

$$n(\boldsymbol{r}) = \int f d^3 \boldsymbol{v} \qquad \langle v_i \rangle = \frac{1}{n} \int v_i f d^3 \boldsymbol{v}$$

Jeans equations

$$n \frac{\partial \langle v_j \rangle}{\partial t} + \sum_{i=1}^{3} n \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -n \frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{3} \frac{\partial n \sigma_{ij}^2}{\partial x_i}$$

(hydrodynamical description)

$$\sigma_{ij}^2 = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$$

Jeans theorem

$$\frac{dI}{dt} = \boldsymbol{v} \cdot \nabla I - \nabla \Phi \cdot \frac{\partial I}{\partial \boldsymbol{v}} = 0$$

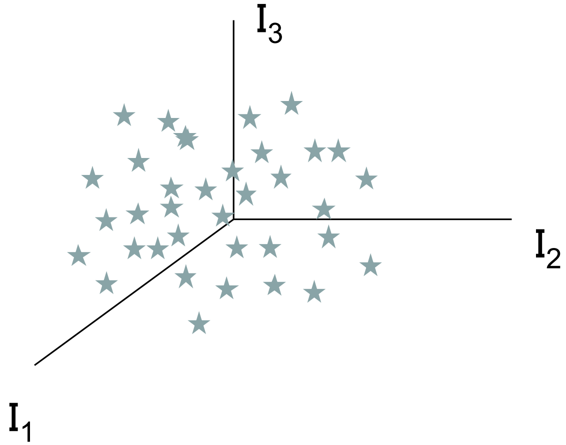
I is a solution to steady-state collisionless Boltzmann eq.

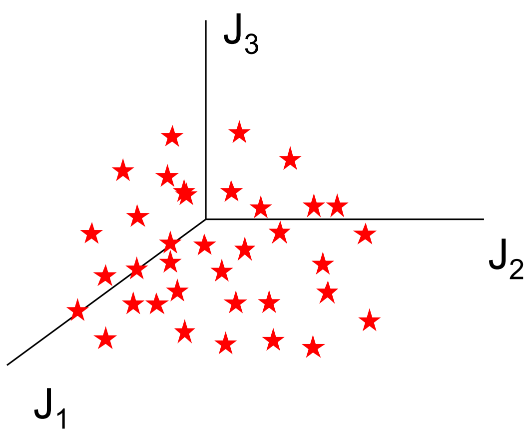
f(I(r,v)): a solution to steady-state collisionless Boltzmann eq.

･Strong Jeans Theorem

Potential Φ allowing only regular orbits
(no resonance among 3 orbital frequencies)
⇒ 3 isolating integrals
⇒ DF depends only these 3 integrals

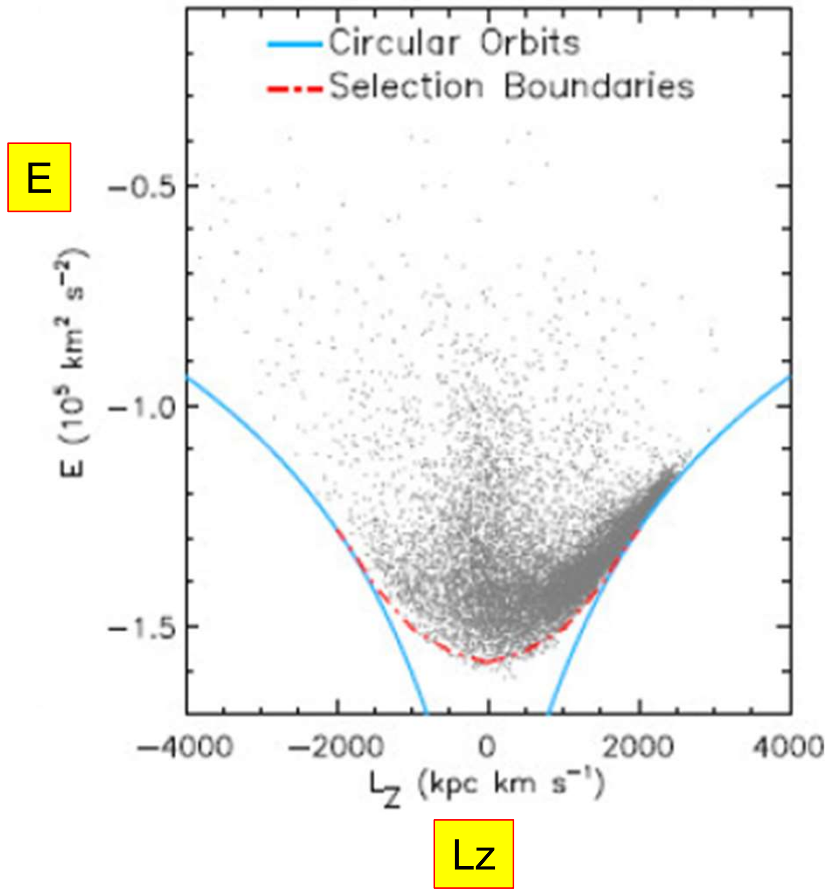
$f(I_1,I_2,I_3)$ $I_1=E$, $I_2=L_z^2/2$ $f(J_1,J_2,J_3)$ $J_i(I_i)$



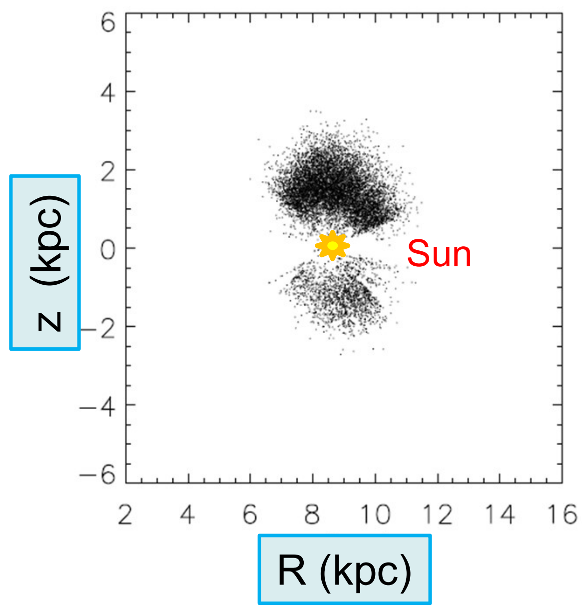


Nearby stars in (E, L_z, I_3) phase space
SDSS-DR7 Calibration Stars + Gaia DR2

Carollo & Chiba 2021

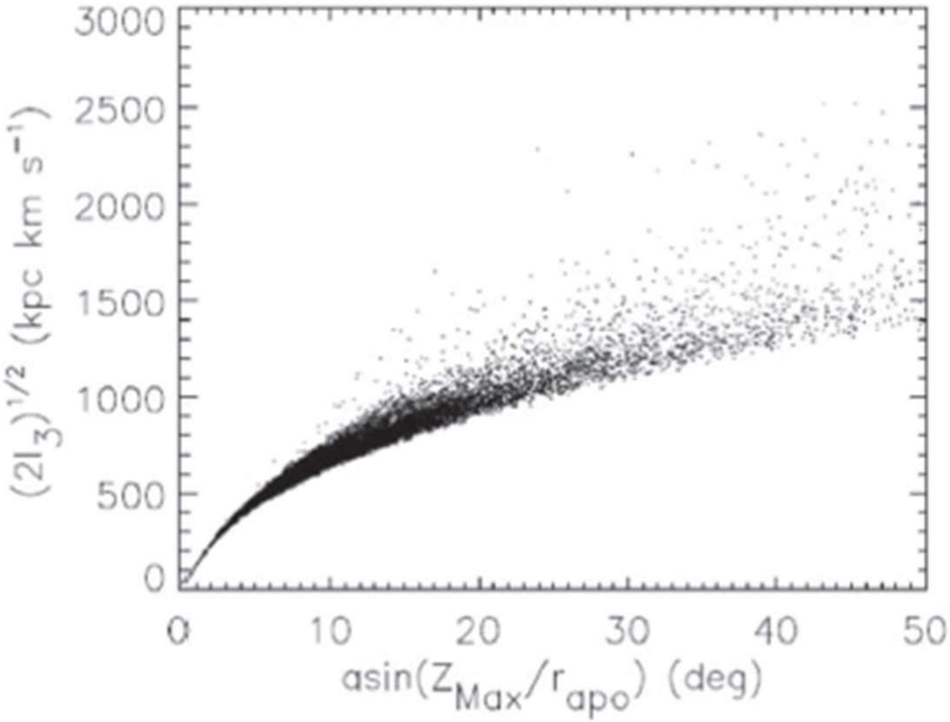


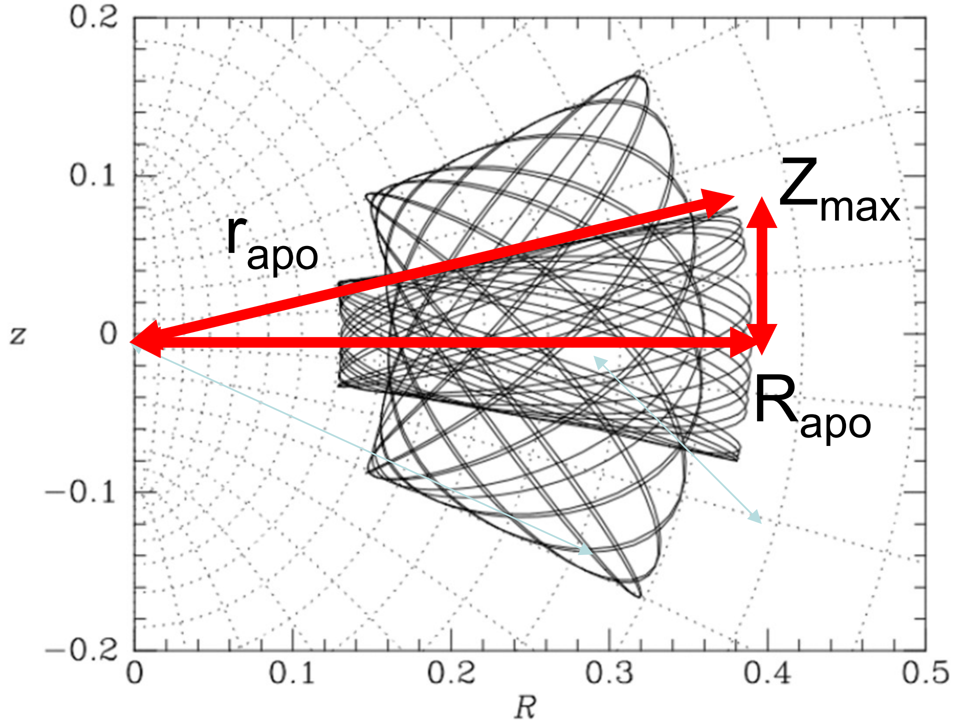
(R,z) distribution
of sample stars



What is the third integral, I_3 ?

$(2I_3)^{1/2}$



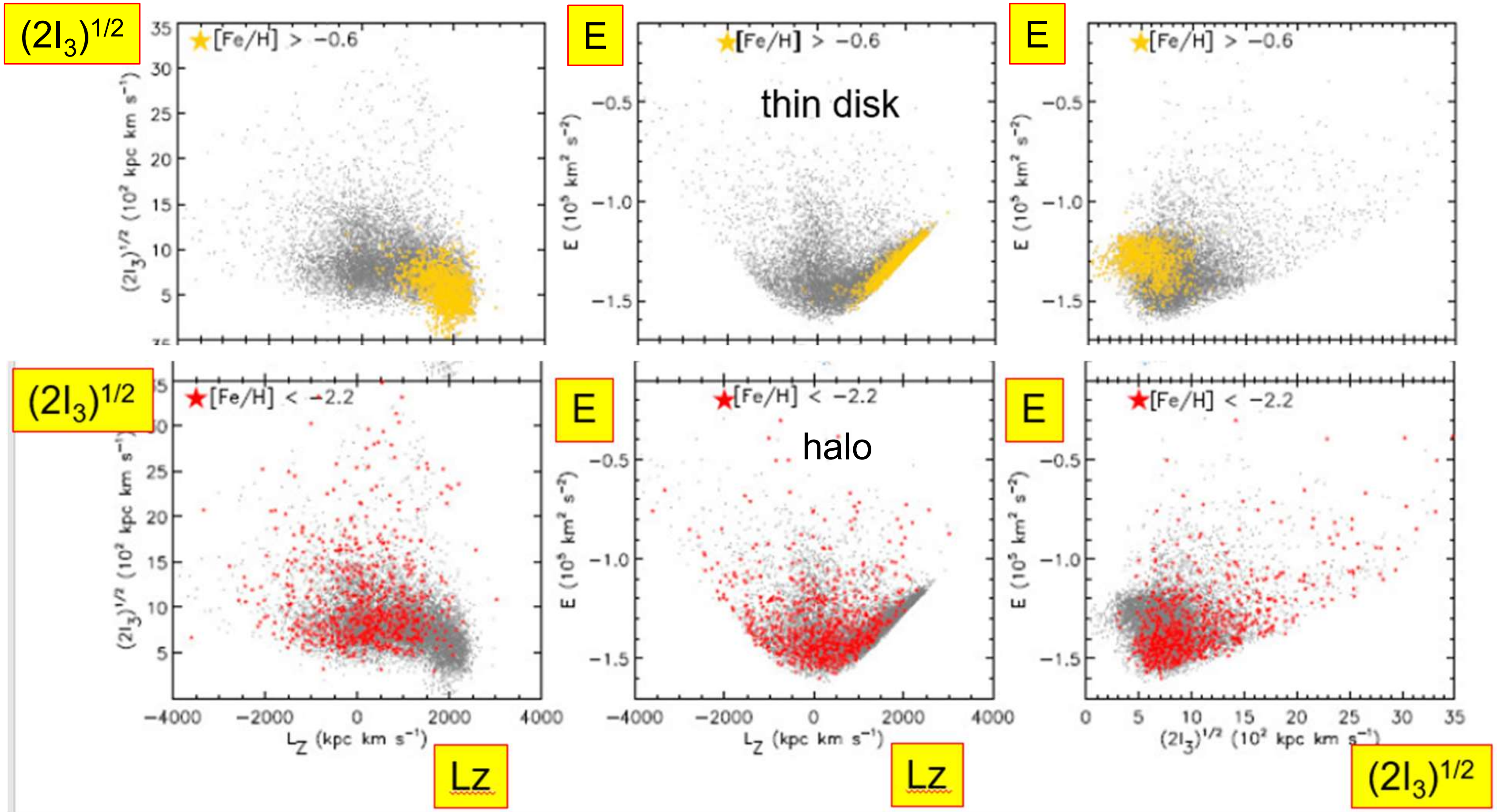


arcsin (Z_max/r_apo) (deg)
~ orbital inclination angle θ

Nearby stars in (E, L_z, I_3) phase space
SDSS-DR7 Calibration Stars + Gaia DR2

(grey: all stars)

Carollo & Chiba 2021



Simple cases for Jeans equations (I)

- Spherical system

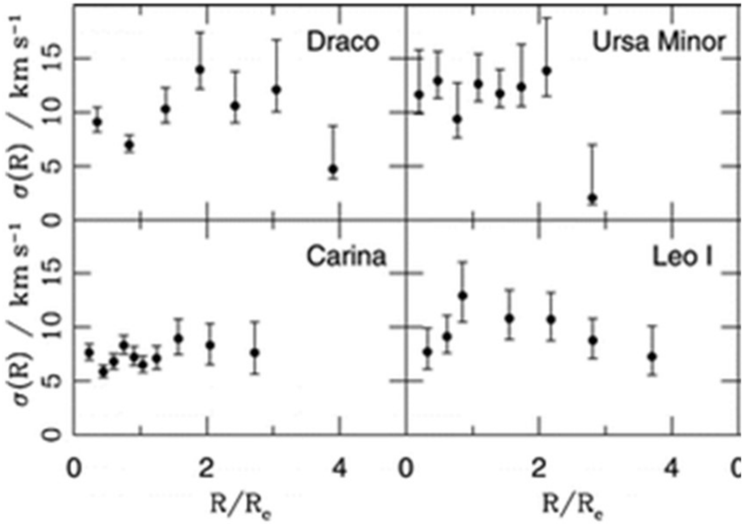
$$\frac{1}{n}\frac{dn\sigma_r^2}{dr} + 2\frac{\beta\sigma_r^2}{r} = -\frac{d\Phi}{dr} = -\frac{GM(r)}{r^2}$$

$$\beta \equiv 1 - (\sigma_\theta^2 + \sigma_\phi^2)/(2\sigma_r^2)$$

$$\beta = const. \Rightarrow \quad n\sigma_r^2 = r^{-2\beta}\int_r^\infty \frac{nGM(r')}{r'^2} r'^{2\beta} dr'$$

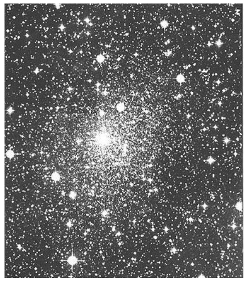
Example: Line-of-sight velocity dispersion profile in MW dwarf satellites

σ_{los}



Leo I

Car



$\sigma_{los}(R)$

$\sigma_r(r)$

M(r)

Dark matter

Simple cases for Jeans equations (II)

- Axisymmetric system

$$\frac{1}{n}\frac{\partial n\langle v_R^2\rangle}{\partial R}+\frac{1}{n}\frac{\partial n\langle v_R v_z\rangle}{\partial z}+\frac{\langle v_R^2\rangle-\langle v_\phi^2\rangle}{R}=-\frac{\partial \Phi}{\partial R}$$

$$\frac{1}{n}\frac{\partial n\langle v_R v_z\rangle}{\partial R}+\frac{1}{n}\frac{\partial n\langle v_z^2\rangle}{\partial z}+\frac{\langle v_R v_z\rangle}{R}=-\frac{\partial \Phi}{\partial z}\ .$$

Example 1: R direction

At z~0 $\partial n/\partial z=0,\ \ \partial\langle v_R v_z\rangle/\partial z=0$

Circular velocity $V_c^2=R\partial\Phi/\partial R$

$$\langle v_\phi\rangle^2=V_c^2-\sigma_R^2\left[\frac{\sigma_\phi^2}{\sigma_R^2}-1-\frac{\partial\ln(n\sigma_R^2)}{\partial\ln R}\right]$$

Asymmetric drift

σ_R^2 is large (old stars) $\langle v_\phi\rangle < V_c$

σ_R^2 is small (young stars) $\langle v_\phi\rangle \simeq V_c$

Example 2: z direction

At z~0 $\partial/\partial z$ >>1

$$\frac{1}{n}\frac{\partial n\sigma_z^2}{\partial z}=-\frac{\partial\Phi}{\partial z}$$

vertical equilibrium

5. Virial theorem

$$\frac{\partial n\langle v_j\rangle}{\partial t} + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(n\langle v_i v_j\rangle) + n\frac{\partial \Phi}{\partial x_j} = 0$$

x mx_k $\rho \equiv nm$

$$\Rightarrow \int x_k \frac{\partial \rho\langle v_j\rangle}{\partial t} d^3x + \int x_k \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\rho\langle v_i v_j\rangle) d^3x + \int \rho x_k \frac{\partial \Phi}{\partial x_j} d^3x = 0$$

$$\boxed{\frac{1}{2}\frac{d^2 I_{jk}}{dt^2} = 2K_{jk} + W_{jk}}$$

$$\begin{cases} I_{jk} \equiv \int \rho x_j x_k d^3x, \\ K_{jk} \equiv \frac{1}{2}\int \rho\langle v_j v_k\rangle d^3x, \\ W_{jk} = W_{kj} \equiv -\int \rho x_k \frac{\partial \Phi}{\partial x_j} d^3x \end{cases}$$

Steady state

$$\boxed{2K + W = 0}$$

$$\begin{cases} K = M\langle v^2\rangle/2 \\ W = -GM^2/R_g \end{cases} \Rightarrow \boxed{\langle v^2\rangle} = \frac{GM}{R_g} \simeq 0.4\frac{GM}{\boxed{R_h}} \Rightarrow M$$

R_g: gravitational radius

R_h: half-mass radius $\simeq 0.4R_g$