一般相対性理論による
ブラックホールの運動

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(二間瀬研M1)

紹介

The Equations of Motion in General Relativity of a Small Charged Black Hole

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電荷ありの小さいブラックホールの一般相対論での運動方程式

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概要

$background\ gravitational\ field$

$ds^2 = \cdots$

$background\ Maxwell\ field$

$F = \cdots$

$Einstein-Maxwell\ vacuum\ field\ equations\ of\ background$

$R_{ab} = 2E_{ab}$

$F^{ab}{}_{;b} = 0$

$blackhole\ as\ perturbation\ of\ background$

$ds^2 = \cdots$

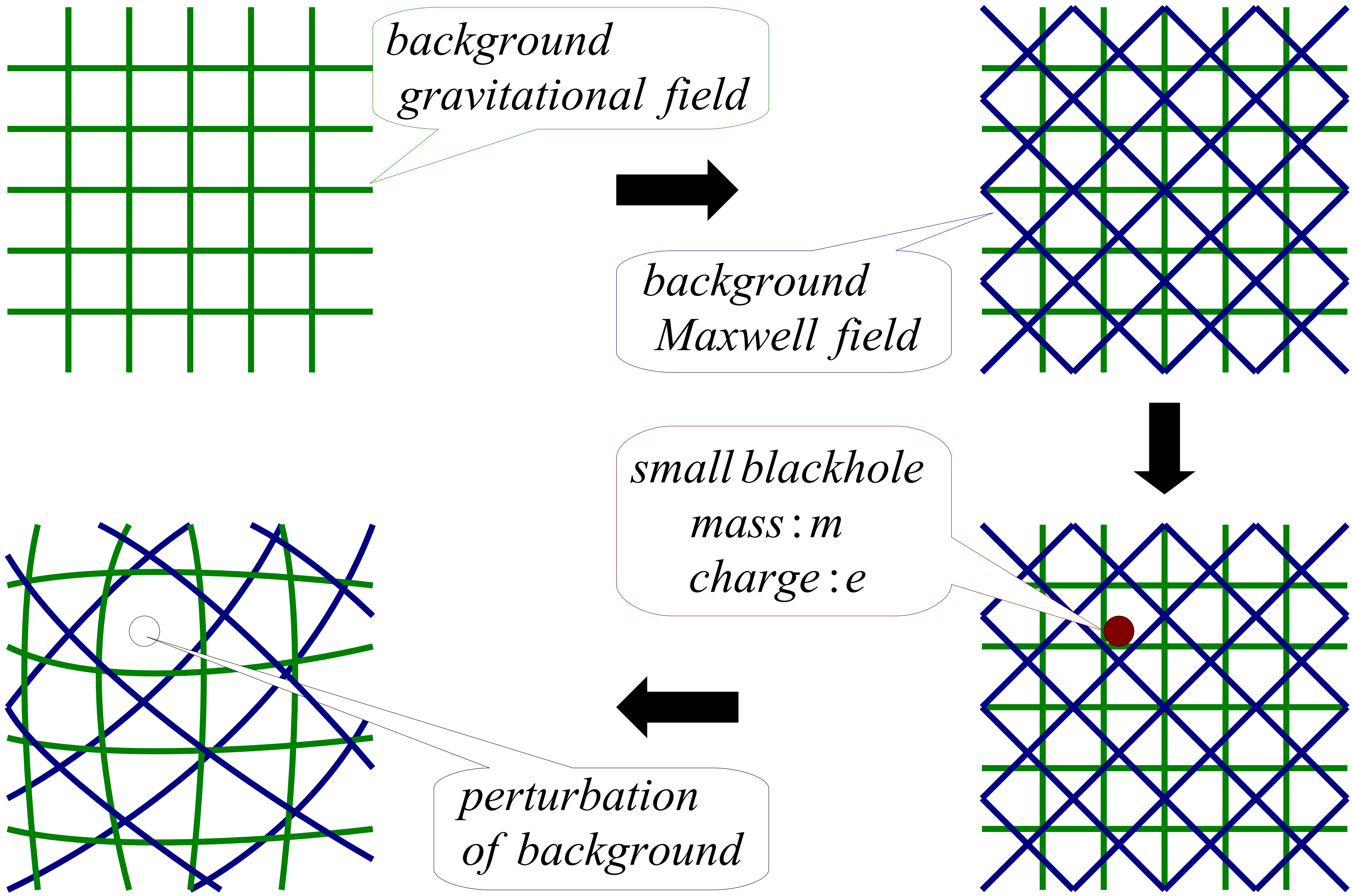
$F = \cdots$

$R_{ab} = 2E_{ab}$

$F^{ab}{}_{;b} = 0$

$\rightarrow equation\ of\ motion : ma = Lorentz + radiation\ reaction + external + tail$

イメージ



background
gravitational field
background
Maxwell field
small blackhole
mass : m
charge : e
perturbation
of background

background

$\mu=0,1,2,3 \quad X^{\mu}=(X^0, X^1, X^2, X^3)$

$g_{\mu\nu}=\eta_{\mu\nu}+O(r^2)$

$X^{\mu}=x^{\mu}(u) \quad at \quad r=0$

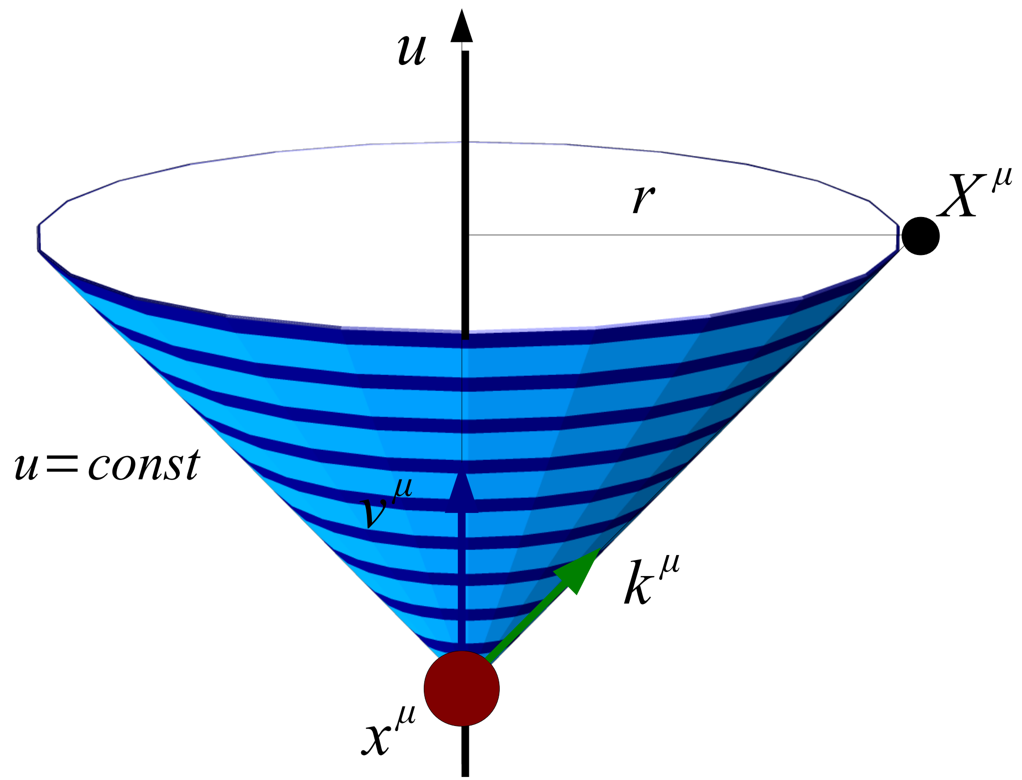
$v^{\mu}(u)=\frac{dx^{\mu}}{du} \quad , \quad a^{\mu}(u)=\frac{dv^{\mu}}{du}$

$r=v_{\mu}(X^{\mu}-x^{\mu}(u))$

$k^{\mu}=\frac{X^{\mu}-x^{\mu}(u)}{r}$

$h_0=a_{\mu}k^{\mu}=\frac{\partial \log P_0}{\partial u}$

$X^{\mu}=x^{\mu}(u)+rk^{\mu}$



$$ds^2=g_{\mu\nu}dX^{\mu}dX^{\nu}$$
$$=-r^2P_0^{-2}(dx^2+dy^2)+dudr+(1-2h_0r)du^2 \quad near \quad r=0$$

Line element of the background

$$ds^2 = -r^2 P_0^{-2}(dx^2 + dy^2) + 2\,du\,dr + (1 - h_0 r)\,du^2$$

$$\downarrow$$

$$ds^2 = -(\theta^1)^2 - (\theta^2)^2 + 2\theta^3\theta^4$$

$$\theta^1 = rP_0^{-1}\,dx$$

$$\theta^2 = rP_0^{-1}\,dy$$

$$\theta^3 = dr + \frac{1}{2}(1 - 2h_0 r)\,du$$

$$\theta^4 = du$$

background space−time near r = 0

general background space−time

$$ds^2 = -(\theta^1)^2 - (\theta^2)^2 + 2\theta^3\theta^4$$

$$\theta^1 = rp^{-1}(e^{\alpha}\cosh\beta\,dx + e^{-\alpha}\sinh\beta\,dy + a\,du)$$

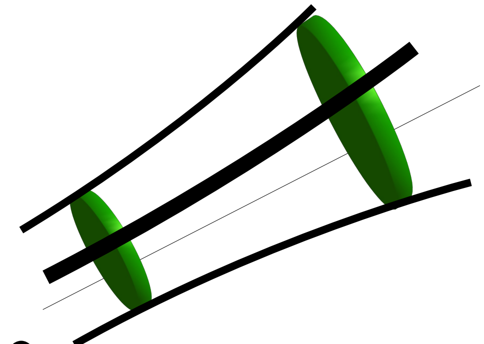
$$\theta^2 = rp^{-1}(e^{\alpha}\sinh\beta\,dx + e^{-\alpha}\cosh\beta\,dy + b\,du)$$

$$\theta^3 = dr + \frac{c}{2}\,du$$

$$\theta^4 = du$$

$p, \alpha, \beta, a, b, c$ は x, y, r, u の関数.
$a = \sum a_i r^i$ のように *powers of r* で表す.
a_i などを様々な条件下で決める.

r → 0 での境界条件



line element

$\rightarrow p_0 = P_0\ ,\ a_0 = b_0 = 0\ ,\ c_0 = 1\ ,\ c_1 = -2h_0$

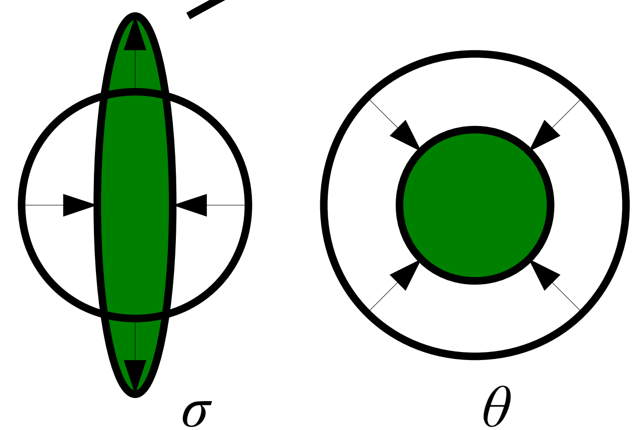
null geodesic congruence

$$\textit{complex shear}: \sigma = \frac{\partial \alpha}{\partial r}\cosh 2\beta + i\frac{\partial \beta}{\partial r} = O(r)$$

$\rightarrow \alpha_i, \beta_i$ は $\alpha_{2,} \beta_2$ から

$$\textit{real expansion}: \theta = \frac{\partial}{\partial r}\log(r p^{-1}) = r^{-1} + O(r)$$

$\rightarrow p_1 = 0$



➡

$p = P_0(1 + q_2 r^2 + q_3 r^3 + \cdots)$

$\alpha = \alpha_2 r^2 + \alpha_3 r^3 + \cdots$

$\beta = \beta_2 r^2 + \beta_3 r^3 + \cdots$

$a = a_1 r + a_2 r^2 + \cdots$

$b = b_1 r + b_2 r^2 + \cdots$

$c = 1 - 2h_0 r + \cdots$

各 r の係数は x, y, u の関数

Background Maxwell field

$potential\ \ one-form$

$A = L\,dx + M\,dy + K\,du\ \ ;\ +W\,dr$ がないのはゲージ変換による

$L = r^2 L_2 + r^3 L_3 + \cdots$

$M = r^2 M_2 + r^3 M_3 + \cdots$

$K = r K_1 + r^2 K_2 + \cdots$

$L_{1,} M_{1,} K_0$ などがあると、F_{ab} が $r \to 0$ で発散してしまい、
$external\ electromagnetic\ field$ としては適さない.

$Maxwell\ \ field$

$$F = dA = \frac{1}{2} F_{ab}\, \theta^a \wedge \theta^b \qquad (tetrad\ indices : a\,, b = 1,2,3,4)$$

$$ds^2 = -(\theta^1)^2 - (\theta^2)^2 + 2\,\theta^3\,\theta^4 = g_{ab}\,\theta^a\,\theta^b \qquad where \qquad g_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Einstein-Maxwell field equations

Einstein's equations

$$\mathcal{E}_{ab} \equiv R_{ab} - 2E_{ab} = 0 \quad \textit{where} \quad E_{ab} = F_{ca}F^{c}{}_{b} - \frac{1}{4}g_{ab}F_{cd}F^{cd}$$

$$\mathcal{E}_{ab} = \sum {}_{(n)}\mathcal{E}_{ab}\, r^{n}$$

$$\mathcal{E}_{11} + \mathcal{E}_{22} = ({}_{(-4)}\mathcal{E}_{11} + {}_{(-4)}\mathcal{E}_{22})r^{-4} + ({}_{(-3)}\mathcal{E}_{11} + {}_{(-3)}\mathcal{E}_{22})r^{-3} + \cdots$$

$$\vdots$$

$$\mathcal{E}_{44} = \cdots$$

Maxwell's equations

$$\mathcal{M}^{a} \equiv F^{ab}{}_{;b} = 0$$

$$\mathcal{M}^{a} = \sum {}_{(n)}\mathcal{M}^{a}\, r^{n}$$

$$\mathcal{M}^{1} = {}_{(-2)}\mathcal{M}^{1} r^{-2} + {}_{(-1)}\mathcal{M}^{1} r^{-1} + \cdots$$

$$\vdots$$

$$\mathcal{M}^{4} = \cdots$$

ポテンシャル1-formの係数

$$ {}_{(1)}\mathscr{M}^{4} = -2P_0^{-2}K_1 + 2\left(\frac{\partial L_2}{\partial x} + \frac{\partial M_2}{\partial y}\right) = 0 $$

$$ {}_{(1)}\mathscr{M}^{3} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)K_1 + 2\left(\frac{\partial L_2}{\partial x} + \frac{\partial M_2}{\partial y}\right) = 0 $$

$$ \rightarrow \underline{\varDelta K_1 + 2K_1 = 0} \quad \textit{where} \quad \varDelta \equiv P_0^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) $$

$$ F = dA = 1/2\, F_{ab}\theta^a \wedge \theta^b = F_{34}\theta^3 \wedge \theta^4 + \cdots $$

$$ \rightarrow F_{34} = K_1 + O(r) $$

$$ \rightarrow \boxed{K_1 = F_{34} = F_{\mu\nu}\varLambda_3^\mu \varLambda_4^\nu = F_{\mu\nu}k^\mu v^\nu \quad on \quad r = 0} \quad \textit{where} \quad dX^\mu = \varLambda_a^\mu \theta^a $$

$$ \rightarrow \underline{\varDelta K_1 = F_{\mu\nu}\varDelta k^\mu v^\nu = -2F_{\mu\nu}(k^\mu - v^\mu)v^\nu = -2F_{\mu\nu}k^\mu v^\nu = -2K_1} $$

$$ \rightarrow \boxed{L_2 = -\frac{1}{2}P_0^{-1}F_{13} = \frac{1}{2}F_{\mu\nu}k^\mu \frac{\partial k^\nu}{\partial x} \quad , \quad M_2 = -\frac{1}{2}P_0^{-1}F_{23} = \frac{1}{2}F_{\mu\nu}k^\mu \frac{\partial k^\nu}{\partial y} \quad on \quad r = 0} $$

Line element の係数

$$ {}_{(0)}\varepsilon_{33} = -8P_0^2(L_2^2+M_2^2)+12q_2=0 $$

$$ \rightarrow q_2 = 2/3\,P_0^2(L_2^2+M_2^2)=\cdots=-1/6\,F^{\lambda}{}_{\mu}F_{\lambda\nu}k^{\mu}k^{\nu} $$

$$ {}_{(0)}\varepsilon_{11} - {}_{(0)}\varepsilon_{22} + 2i\,{}_{(0)}\varepsilon_{12} = -12(\alpha_2+i\beta_2) - 6\frac{\partial}{\partial\bar{\zeta}}\left(a_1+ib_1+4P_0^2\frac{\partial q_2}{\partial\bar{\zeta}}\right)=0 $$

$$ {}_{(0)}\varepsilon_{13}+i\,{}_{(0)}\varepsilon_{23} = -2\mathrm{P}_0^{-1}\left(a_1+ib_1+4P_0^2\frac{\partial q_2}{\partial\bar{\zeta}}\right)+4P_0^3\frac{\partial}{\partial\zeta}\left[P_0^{-2}(\alpha_2+i\beta_2)\right]=0 $$

$$ \rightarrow \frac{\partial}{\partial\bar{\zeta}}\left\{P_0^4\frac{\partial}{\partial\zeta}\left[P_0^{-2}(\alpha_2+i\beta_2)\right]\right\} = -(\alpha_2+i\beta_2) \qquad where \qquad \zeta\equiv x+iy $$

⋮

$$ \alpha_2=\frac{1}{6}P_0^2C_{\mu\nu\rho\sigma}k^{\mu}\frac{\partial k^{\nu}}{\partial x}k^{\rho}\frac{\partial k^{\sigma}}{\partial x} \quad , \quad \beta_2=\frac{1}{6}P_0^2C_{\mu\nu\rho\sigma}k^{\mu}\frac{\partial k^{\nu}}{\partial x}k^{\rho}\frac{\partial k^{\sigma}}{\partial y} $$

$$ a_1=\frac{2}{3}P_0^2\left(C_{\mu\nu\rho\sigma}k^{\mu}v^{\nu}k^{\rho}\frac{\partial k^{\sigma}}{\partial x}+F^{\lambda}{}_{\mu}F_{\lambda\nu}k^{\mu}\frac{\partial k^{\nu}}{\partial x}\right) \quad , $$

$$ b_1=\frac{2}{3}P_0^2\left(C_{\mu\nu\rho\sigma}k^{\mu}v^{\nu}k^{\rho}\frac{\partial k^{\sigma}}{\partial y}+F^{\lambda}{}_{\mu}F_{\lambda\nu}k^{\mu}\frac{\partial k^{\nu}}{\partial y}\right) $$

Black hole as perturbation

$mass : m = O_1 , charge : e = O_1$ のブラックホールを摂動として扱い、
m, e の高次は落とす.
$r, m, e \to 0 \, (m/r, e/r$ は有限 $)$ の境界条件
○重力場 $\cdots$ *Reissner − Nordstrom blackhole*
$$ds^2 = (1 - 2\mathrm{m}/r + e^2/r^2) du^2 + O(r)$$
○電磁場 $\cdots$ *Liénard − Wiechert potential*
$$A = e(r^{-1} - h_0) du + O(r)$$

$$p = \hat{P}_0 (1 + q_2 r^2 + q_3 r^3 + \cdots)$$
$$\alpha = \alpha_2 r^2 + \alpha_3 r^3 + \cdots$$
$$\beta = \beta_2 r^2 + \beta_3 r^3 + \cdots$$
$$a = a_{-1} r^{-1} + a_0 + a_1 r + a_2 r^2 + \cdots$$
$$b = b_{-1} r^{-1} + b_0 + b_1 r + b_2 r^2 + \cdots$$
$$c = e^2 r^{-2} - 2(m + 2 f_{-1}) r^{-1} + c_0 + c_1 r + \cdots$$

$$L = L_0 + r^2 L_2 + r^3 L_3 + \cdots$$
$$M = M_0 + r^2 M_2 + r^3 M_3 + \cdots$$
$$K = r^{-1}(e + K_{-1}) - e h_0 + K_0 + r K_1 + \cdots$$

background には無い O_1 O_2
他は *background* $+ O_1$

ブラックホールからの波

$u = const$

$ds_0^2 = -r^2 \hat{P}_0^{-2}(dx^2 + dy^2) + O(r^4)$

$\hat{P}_0 = P_0(1 + Q_1 + Q_2 + O_3)$

$Q_1 = O_1 \quad , \quad Q_2 = O_2$

↓

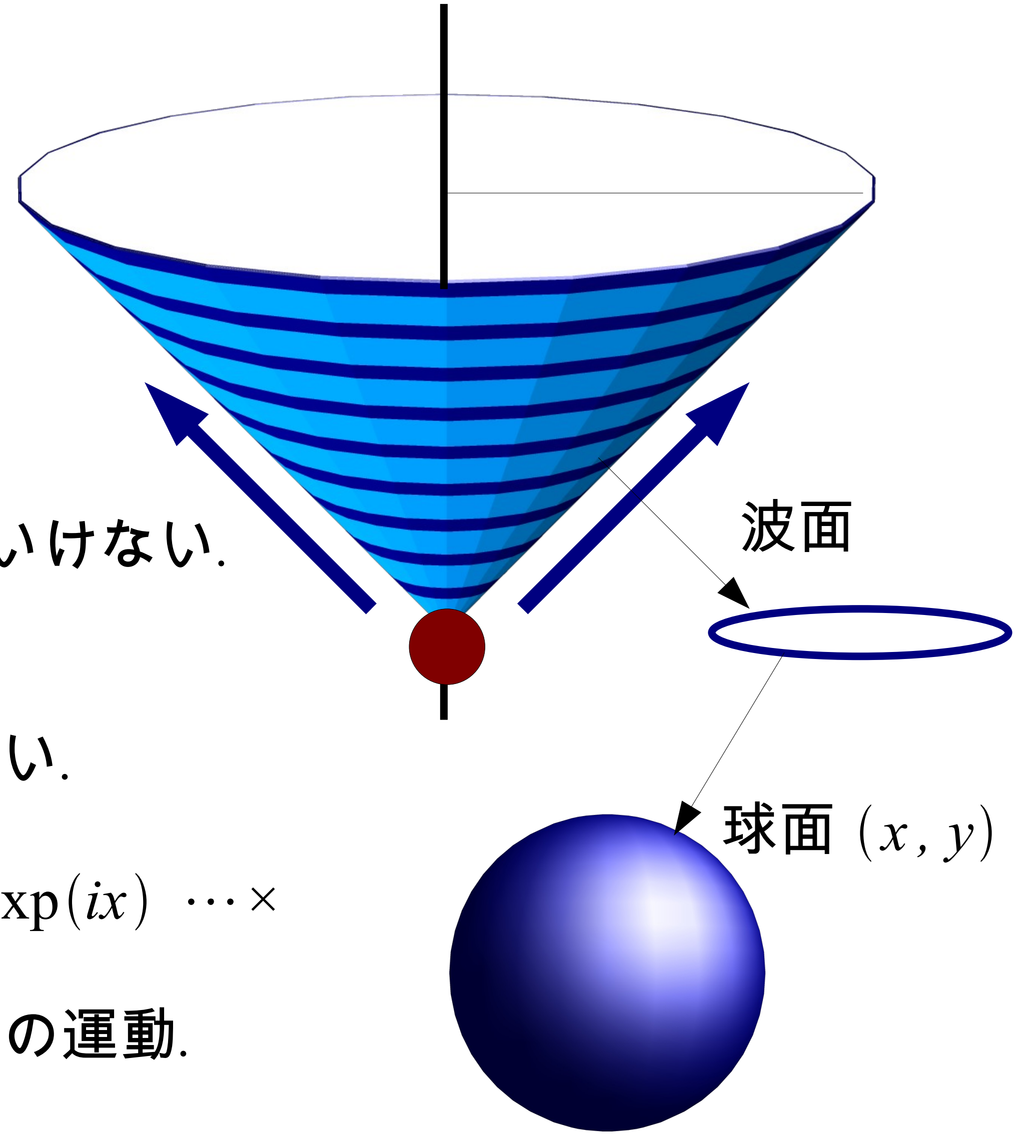
Einstein−Maxwell vacuum
*field equations*を解くときに、
*directional singularities*が現れてはいけない.

↓

２次元球面の摂動 Q_1, Q_2が、
あらゆる x, yで発散してはいけない.
例えば

$Q(x) = \exp(\sqrt{2}ix) \quad \cdots \bigcirc \qquad Q(x) = x\exp(ix) \quad \cdots \times$

この球面の動きがブラックホールの運動.



運動方程式への道

$$
\begin{aligned}
{}_{(-2)}\varepsilon_{44} = & -8e^2 F^p{}_i F_{pj} k^i v^j - 16 e^2 (F_{ij} k^i v^j)^2 \\
& + \frac{1}{2}\Delta c_0 - \frac{1}{2} c_0 P_0^2 \left[\frac{\partial}{\partial x}\left(P_0^{-2} a_{-1}\right) + \frac{\partial}{\partial y}\left(P_0^{-2} b_{-1}\right) \right] - \frac{1}{2} a_{-1} \frac{\partial c_0}{\partial x} - \frac{1}{2} b_{-1} \frac{\partial c_0}{\partial y} \\
& + 3m P_0^2 \left[\frac{\partial}{\partial x}\left(P_0^{-2} a_0\right) + \frac{\partial}{\partial y}\left(P_0^{-2} b_0\right) \right] - 4 \frac{\partial f_{-1}}{\partial u} + 12 f_{-1} h_0 + 6 m h_0 \\
& - 4 e^2 P_0^2 L_2 \frac{\partial K_1}{\partial x} - 4 e^2 P_0^2 M_2 \frac{\partial K_1}{\partial y} - 2 P_0^2 \left[\left(\frac{\partial K_0}{\partial x} \right)^2 + \left(\frac{\partial K_0}{\partial y} \right)^2 \right] \\
& + 8 m P_0^2 L_2 \frac{\partial K_0}{\partial x} + 8 m P_0^2 M_2 \frac{\partial K_0}{\partial y} - 4 P_0^2 L_2 \frac{\partial K_{-1}}{\partial x} - 4 P_0^2 M_2 \frac{\partial K_{-1}}{\partial y} \\
& - 4 P_0^2 \frac{\partial K_1}{\partial x} \frac{\partial K_{-1}}{\partial x} - 4 P_0^2 \frac{\partial K_1}{\partial y} \frac{\partial K_{-1}}{\partial y} - \frac{5}{2} e^2 P_0^2 \left[\frac{\partial}{\partial x}\left(P_0^{-2} a_1\right) + \frac{\partial}{\partial y}\left(P_0^{-2} b_1\right) \right] \\
& + O_3
\end{aligned}
$$

各項の $c_{0,} f_{-1,} a_{-1,} a_{0,} a_{1,} b_{-1,} b_{0,} b_{1,} L_{2,} M_{2,} K_{-1,} K_{0,} K_1$ は他の方程式から得る.

オーダー1まで

$$ {}_{(-2)}\varepsilon_{44} = \frac{1}{2}\Delta(\Delta Q_1 + 2Q_1) + 6mh_0 - 6eF_{ij}k^i v^j + O_2 $$

$$ \rightarrow \Delta(\Delta Q_1 + 2Q_1) = 6ma_i\Delta p^i - 6eF_{ij}\Delta p^i v^j + O_2 \quad where \quad p^i = k^i - v^i = -\frac{1}{2}\Delta p^i $$

$$ \rightarrow \Delta Q_1 + 2Q_1 = 6ma_i p^i - 6eF_{ij}p^i v^i + A(u) + O_2 $$

簡単のために

$$ \Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ , \ \hat{Q}_1 \equiv Q_1 - \frac{1}{2}A(u) \ , \ 4\exp[i\sqrt{2}(x+y)] = 6ma_i p^i - 6eF_{ij}p^i v^i $$

とすると、

$$ \Delta\hat{Q}_1 = -2\hat{Q}_1 + 4\exp[i\sqrt{2}(x+y)] $$

$$ \rightarrow Q_1 = (1 + x^2 - 2xy + y^2)\exp[i\sqrt{2}(x+y)] + \frac{1}{2}A(u) \quad \cdots x \rightarrow \infty, y \rightarrow -\infty $$ で発散

Q_1 に *directional singularities* が現れないために

$$ ma_i = eF_{ij}v^j + O_2 \quad \cdots equation\ of\ motion $$

$$ \rightarrow Q_1 = \exp[i\sqrt{2}(x+y)] + \frac{1}{2}A(u) \quad \cdots $$ 発散しない

オーダー2まで

Q_1 は $l=0,1$ の *spherical harmonics* の和で表された.

$l=0$ は球の半径を、$l=1$ は球の位置を少し変えるだけ. $\to Q_1=0$

Q_2 に関して ${}_{(-2)}\varepsilon_{44}$ を計算すると、

$$-\frac{1}{2}\Delta(\Delta Q_2+2Q_2)=A_0+A_1+A_2+O_3 \quad where \quad A_l: spherical\ harmonics\ of\ l$$

directional singularities なく積分するためには、

$$A_0=O_3 \to \dot{G}=-\frac{4}{3}e^2F^p{}_iF_{pj}v^iv^j \to \boxed{G=-\int_{-\infty}^{u}\frac{4}{3}e^2F^p{}_iF_{pj}v^iv^j\,du}$$

$\Delta A_1=-2A_1$, $\Delta A_2=-6A_2$ を用いて積分できて、

$$\to \Delta Q_2+2Q_2=A_1+\frac{1}{3}A_2+O_3=-\frac{1}{12}(\Delta A_2+2A_2)+A_1+O_3$$

directional singularities がない Q_2 を得るためには、

$A_1=O_3$

$$\to m a_i=eF_{ij}v^j+\frac{2}{3}e^2h_i^k\dot{a}_k+\frac{4}{3}e^2h_i^kF^p{}_kF_{pj}v^j$$

$$+\frac{4}{3}me\dot{F}_{ij}v^j-2Ga_i+2e^2K(u)F_{ij}v^j+U(u)c_i+V(u)d_i+O_3$$

$$c_i p^i \equiv \frac{\partial \log P_0}{\partial x},\quad d_i p^i \equiv \frac{\partial \log P_0}{\partial y}$$

もう少し

新しい文字やローレンツ変換などを使ってシンプルにすると、

$$m a_i = e F_{ij} v^j + \frac{2}{3} e^2 h_i^k \dot{a}_k + \frac{4}{3} e^2 h_i^k F^p{}_k F_{pj} v^j + T_i + O_3$$

右辺はそれぞれ

1st : *Lorentz force*

2nd : *radiation reaction*

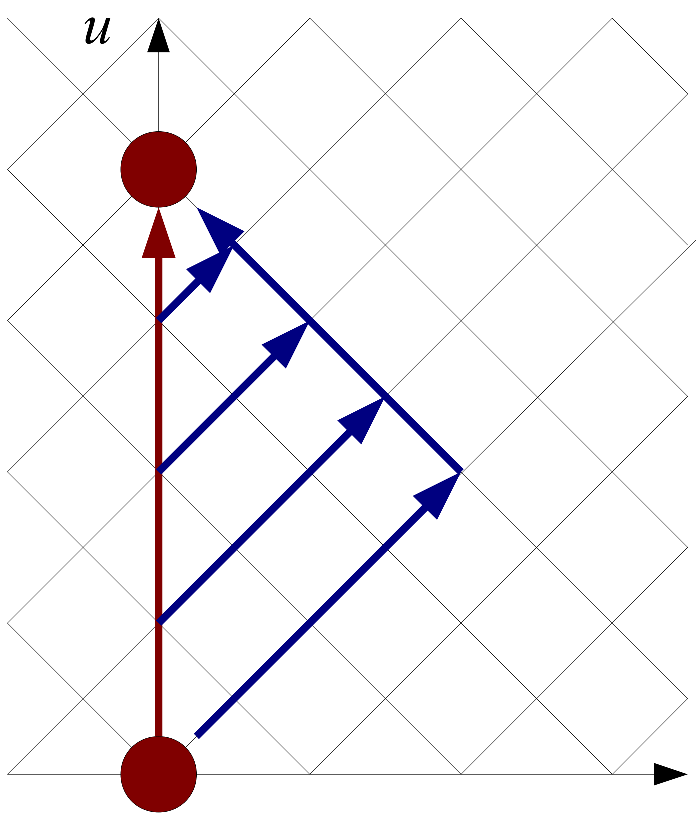
3rd : *external force*

4th : *tail term*

$$T_i = \frac{e}{m} \left(\omega_k{}^j F_{ji} - \omega_i{}^j F_{jk} - 2 G F_{ik} \right) v^k$$

$$= \int_{-\infty}^{u} du [\cdots]$$

tail term は無限の過去から現在までの時間積分であり、過去の影響が時空を巡って現在に力を及ぼしている.



さいご

- 先行研究との比較
- より高次の導出
- ここで得られた時空を他に利用
- 初期の問題設定の変更
  例:スピンするブラックホール

以上